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ABSTRACT

This sixth module in the series for preservice elementary teachers develops the following concepts of diagnosis in mathematics: diagnostic techniques and instruments; the identification of specific learner behaviors; the concrete, semi-concrete, and abstract continuum aspect of learning mathematics concepts and skills; Piagetian developmental stages for conservation of number, length, volume, substance, and area; learner error patterns; and remedial procedures. Objectives, prerequisites, and pre- and post-assessment procedures for this module are specified, and sixteen instructional activities to be completed by the preservice teachers are listed. Fifteen reading handouts are included with this module. (DT)

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OBJECTIVES:

At the completion of module study, the Intern will be able to...

1. (Knowledge) DEFINE and DESCRIBE the following concepts of diagnosis:

   a. Means-Oriented Diagnosis: (which emphasizes diagnostic techniques, instruments, or procedures to be used BEFORE and DURING an instructional sequence.)

   b. Learner-Goal-Oriented Diagnosis: (which emphasizes identifying the SPECIFIC LEARNER' BEHAVIORS which are in evidence or are lacking AT THE BEGINNING OF AN INSTRUCTIONAL SEQUENCE.)

   c. The C/SC/A Continuum Concept of Diagnosis: (which emphasizes Concrete; Semi-Concrete; and Abstract aspects of learning math concepts and skills.)

2. (Knowledge-Performance-Consequence) DEFINE and DIAGNOSE by DEMONSTRATING with learners the following Prayetian Concepts to determine the learners' developmental stages of readiness for these conservations in math:

   a. Number
   b. Length
   c. Volume
   d. Substance
   e. Area

   (NOTE: UTILIZE the attached assessment sheet "Diagnosing Conservation Concepts.")

3. (Performance-Consequence) DIAGNOSE learner "error-patterns" in math concepts and PRESCRIBE and DEMONSTRATE appropriate remediational procedures.

   (NOTE: UTILIZE the attached assessment sheet "Learner Error-Pattern Diagnosis.")

PRE-REQUISITE:

Completion of modules CMM:006:01(ISU)-CMM:006:05(ISU)

PRE-ASSESSMENT:

1. COMPLETE the Pre-Assessment of the Knowledge portions of the objectives and CHECK with the Team Leader. If the minimal performance level is not achieved on Pre-Assessment, then the Intern should select or design appropriate Instructional Activities for study prior to attempting the Post-Assessment.

2. Since a portion of Objectives 2 and 3 require Intern performance demonstrations in diagnosis and consequence objectives (learner outcomes), these objectives may be less likely to be Pre-Assessed by the Intern prior to module study. If such demonstrations are felt possible, then the Intern should submit evidences to the Team Leader.

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INSTRUCTIONAL ACTIVITIES:

1. READ Lloyd Scott: "The Changing View of Learning" (In: Trends in Elementary School Mathematics, Rand McNally & Co., 1966, pp. 9-16.) Scott identifies how learners develop concepts (ala Piaget, Bruner, Dienes, etc.) and indicates the need for increased emphasis upon determining learners' appropriate levels of conceptualization in math--a vital aspect of diagnosis.

2. READ E.E. Biggs: "Research in Children's Method of Learning" (In: Mathematics in Primary Schools, Curr. Bulletin No. 1, Schools Council, London, England, 1965, pp. 5-9.) Biggs reviews Piaget's research which emphasizes providing learners with "activity-oriented learning experiences." This emphasis provides the focus for using Piagetian tests as means to Diagnose learners' concept development.

3. READ Dienes and Golding: "The Classroom Situation" (In: Z.P. Dienes and E.W. Golding: Learning Logic, Logical Games, Herder and Herder, 1966, pp. 10-14.) This is an additional emphasis upon an "activity-centered approach to learning." This article describes a "diagnosis-in-the-process" type of orientation with learners.

4. READ Cecil R. Trueblood: "A Model for Using Diagnosis in Individualizing Mathematics Instruction in the Elementary Classroom" (In: Arithmetic Teacher, Nov. 1971, pp. 505-511.) The "means-oriented" and "goal-oriented" models of diagnosis are contrasted in this article. They particularly modelize the "goal-referenced model" for diagnosis and individualization.

5. READ G. Schlinsog: "Managing Mathematics in the Early Education of Children" (In: Schmicke and Arnold: Math is a Verb, Dryden Press, 1971) This brief article describes concepts and skills in early math development and describes some features of programs which are appropriate for very young learners--especially emphasizing "active engagement" of learners. Many diagnostic ideas can be gleaned from this brief article.

6. READ A.S. Picard: "Piaget's Theory of Development with Implications for Teaching Elementary School Mathematics" (In: School Science and Mathematics, April 1969, pp. 275-80.) This article defines the concept of "operations" so basic to Piaget's position and states a number of suggestions for Diagnosing children's stages of development of various operations.

7. READ M. Smart: "What Piaget suggests to classroom teachers." This article describes the nature of Diagnostic experiments to conduct with young learners and specifies ways for analyzing learners' responses in obtaining Diagnostic data.

8. STUDY the following Piagetian experiments illustrated in HANDOUT: "Testing Readiness:"

   Experiment 1: Conservation of number
   Experiment 2: Conservation of length
   Experiment 3: Conservation of volume
   Experiment 4: Conservation of substance
   Experiment 5: Conservation of area

INSTRUCTIONAL ACTIVITIES: (cont.)

8. (cont.) The Intern is encouraged to utilize one or more examples of each of these "conservations" for checking young learners' "readinesses" for stages of developmental growth. Simple materials and "starter questions" can be readily prepared from the information provided.

9. STUDY-READ T.A. West: "Diagnosing Pupil Errors: Looking for Patterns" (In: Arithmetic Teacher, Nov. 1971, pp. 467-469.) This article identifies "conceptual" and "careless" error types--conceptual errors are detected by seeing patterns of errors in a learner's work and careless errors are only sporadic errors and may be detected if learner is taught one or more procedures for "checking" his work.

10. READ R. Underhill: "Diagnosis: Reflections on the C/SA/A Continuum" (In: R. Underhill: Teaching Elementary School Math, Professional Supplement, Charles E. Merrill, 1972, pp. 99-103.) At any level of instruction, learners may need to have learning experience at a concrete level (manipulative experiences with physical objects), at a semi-abstract level (pictorial representations such as drawings or pictures to represent physical objects), or at an abstract level (symbols such as numerals and math symbols).

    (NOTE: It is critically important to note that concrete or manipulative experiences are not reserved for young learners. Learners at any level often need to "modelize or experience" math operations and processes at the concrete level before they represent their concepts at the semi-abstract or abstract levels.)

11. TRY COMPLETING Welch & Edwards': "Test of Arithmetic Principles: Elementary Form" (In: Bulletin of School of Education, Indiana University, Volume 41, No. 5. Sept. 1965.) No answers are given so the Intern may need to verify some responses to determine the correct responses; however, this is a test of "principles" and answers are of much less interest than the thinking processes that learners go through while "working out answers." This test can also serve as a "pool" for diagnostic items for short diagnostic instruments or procedures which the Intern might develop to check learners on certain principles.

12. STUDY: Maertens and Dunlap: "Diagnostic testing and teaching through constructing and using a hierarchy of mathematical tasks" (In: Maertens & Dunlap: Meeting Individual Needs in Elementary Math, Dryden Press, 1973, p. 1-13.) This material describes a procedure for breaking down a mathematical task into its various subordinate and interrelated tasks. This procedure is particularly valuable for a teacher to go through prior to introducing some new skill or concept, since it reveals those pre-requisite tasks which learners will need to demonstrate to be successful with the terminal task. The Intern is urged to "try to develop" such a hierarchy of tasks for some concept they are teaching.

13. READ Maertens & Dunlap: "Using the Child's Environment to Diagnose Problem-Solving Competency" (In: Maertens & Dunlap: Meeting Individual Needs in Elementary Math, Dryden Press, 1973, pp. 45-46.) This brief presentation gives an example of the use of "real-life" problems for assessing learners' problem-solving competence.

INSTRUCTIONAL ACTIVITIES: (cont.)

14. STUDY and WORK THROUGH the Handout "Constructing a Diagnostic test in Math." This Handout provides examples of assessment items and procedures for checking learners computation, manipulation, and problem-solving skills in terms of math tasks performed.

15. STUDY Ashlock: Error Patterns in Computation: A Semi-Programmed Approach to:

    (1) Examine error patterns in computation.
    (2) Your ability to identify the nature of an error pattern in computation.
    (3) Determine remediational approaches designed to correct error patterns in computation.

16. DESIGN and CARRY OUT your own Instructional Activities designed to demonstrate the objective on Diagnosis.

POST-ASSESSMENT:

1. Demonstrate to Team Leader during a conference your "knowledge" objectives. The Team Leader should verify the results of your conference.

2. Attach a description of your evidence of the demonstration of the performance and consequence objectives. The Team Leader should verify the evidence.

COMPLETE the Post-Assessment sheet and submit to the Instructor.

REMEDIATION:

Determine in consort with the Team Leader or Instructor.

A diagnostic test should indicate the precise level of understanding of a learner as related to any math task. It should indicate whether the child's learning is at the rate level, concept level, or principles level and whether he is able to use these principles in the solution of problems.

Below are some "suggested" types of assessment items--some to check computation, some to check manipulation, and some to check problem-solving.

As you study each item, identify the mathematical task tested by each item. Write your responses in the space provided and then check again the KEY.

MANIPULATION-TYPE ASSESSMENT ITEMS:

(NOTE: The Teacher or Helper physically performs the action on objects in each situation and gives oral directions such as: "I am going to use objects to show an action. Write a number sentence to show what I have done." With a young learner, the directions might request the learner to "simply verbalize" the process in some way.)

PART I:

ITEMS	YOUR INDICATION OF THE MATHEMATICAL TASKS INVOLVED:
1. T: Removes 6 objects from a set of 14 objects.	
2. T: Joins a set of 6 objects with a set of 4 objects.	
3. T: Separates a set of 12 objects into 3 sets of 4 objects.	
4. T: Joins 2 sets of 3 objects, a set of 2 objects, and a set of 5 objects.	

PART II: (Teacher or Helper gives the Learner 25 objects, shows a number sentence to the Learner, and gives these directions: "Using these objects, show what the number sentence means and find the answer." A young learner may be given fewer than 25 objects but would still be expected to show what the number sentence meant and demonstrate the answer.)

ITEMS	YOUR . . . TAKS:
6. T: Writes □ = 6 + 5	
7. T: Writes 2 x 4 = □	
8. T: Writes □ = 12 - 7	
9. T: Writes 8 + 4 = □	
10. T: Writes 12 - □ = 3	

PART III: (Teacher or Helper will show actions on objects and give directions for each item.)

ITEMS	YOUR . . . TASKS:
11. T: Groups 23 objects into 2 groups of 10 and 3 ones. State: "Using these objects, show what the number sentence means and find the answer to the exercise 23 - 9 = □." (T. writes the number sentence.)	
12. T: Groups 25 objects into 2 sets of 10 and 5 ones and gives an additional 8 objects to the Learner. State: "Using these objects, show what the number sentence means and find the answer to the exercise 25 + 8 = □ ." (T. writes the number sentence.)	
13. T: Place 24 ungrouped objects before the Learner. State: "Put the objects into groups of 10 and write a numeral showing the number of 10's and 1's."	
14. T: Give the Learner 32 objects grouped into 3 sets of 10 and 1 set of 2. State: "regroup the objects so that there is one less group of 10. Write a number sentence to show what you did."	

15. T: Puts out sets of 2, 6, and 3 objects. Joins the sets of 2 and 6 objects. Joins sets of 6 objects and 3 objects and then joins a set of 2 objects with the resulting set. State: "Write a number sentence to show what I have done."

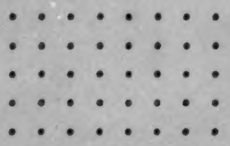
16. T: Give child 25 objects. State: "Using the objects, show how to find the answer to the number sentence 8 + 0 = ☐ ."

17. T: Give the Learner 25 objects. State: "Using the objects, show how to find the answer to the number sentence 9 - 0 = ☐ ."

18. T: Give 25 objects to child. State: "Using these objects, show how to find the answer to the number sentence 5 + 0 = ☐ ."

19. T: Give 25 objects to child. State: "Use these objects to show how to find the answer to the number sentence 3 x 10 = ☐ ."

20. T: Give the child the following ARRAY on a piece of paper:



Write or give this problem to child:
5 x 8 = 5 x (5 + 3)
State: "Illustrate the number sentence by drawing lines on the array."
(NOTE: You could use an array board and rubber bands. . .)

21. T: Give 25 objects to child and write the equation 6 x 3 = 18 for the child. State: "Use these objects to show that multiplication is opposite of division."

COMPUTATION-TYPE ASSESSMENT ITEMS:

(**NOTE:** Normally these kinds of assessment items are presented on a worksheet. The instructions given are: "Write the answer to each problem.")

PART I:

ITEMS	YOUR INDICATION OF MATH TASK INVOLVED:
1. $1289 + \square = 1289$	
2. $2600 + 40 + 4000 + 9 + 300 = \square$	
3. $\begin{array}{r} 7 \\ 45 \\ 273 \\ \underline{95} \end{array}$	
4. $\square = 4596 - 0$	
5. $\begin{array}{r} 502 \\ \underline{-\ 357} \end{array}$	
6. $7869 \times \square = 0$	
7. $10 \times 7896 = \square$ $1289 \times \square = 1,289,000$ $\square \times 89 = 8900$	
8. $\begin{array}{r} 405 \\ \underline{\times\ 213} \end{array}$	
9. $\square = 4591 = 4591$	
10. $7800 \div 1000 = \square$ $790 \div \square = 79$ $\square \div 100 = 89$	
11. $24\overline{)\,3567}$	

PART II: (NOTE: Directions for each of the following will be given for each item.)

ITEMS	YOUR INDICATION OF MATH TASK INVOLVED:
12. T: "Write a number sentence showing 12 ones and 3 tens re-written as a numeral."	
13. T: "What is the place value occupied by 5 in the numeral 75,691?"	
14. T: "Write a number sentence to show a regrouping from hundreds to tens for the numeral 475."	
15. T: "Multiply the 3 factors (4, 5, 2) by grouping them in 2 different ways."	
16. T: "Finish the number sentence to find an answer: 5 x 7 = 5 x (4 + 3) = ☐ + ☐ = ☐."	
17. T: "Finish the number sentence to find an answer: 24 ÷ 3 = (12 + 9 + 3) ÷ 3 = ☐ + ☐ + ☐ = ☐ ."	

PROBLEM-SOLVING-TYPE ASSESSMENT ITEMS:

(NOTE: Present the story problems to the child on a sheet of paper or orally or both. Child may work out the problem on scratch paper and state the answer. You are urged to watch the child as problem-solving is going on.)

ITEMS:	LIST WHAT 6 TASKS THE LEARNER MUST PERFORM IN EACH OF THESE PROBLEMS
1. "6 sisters had 191 seashells in a box. Susan, Janet, and Carol are 3 of the sisters. Susan had 59 shells and Janet had 61 shells. Carol used 67 of the shells on place cards for a party. How many seashells WERE IN THE BOX during the party?"	

ITEMS	LIST WHAT 6 TASKS THE LEARNER MUST PERFORM IN EACH OF THESE PROBLEMS
2. "Kay has a stamp book containing 26 pages. She collects stamps as a hobby. On 12 pages in the stamp book, Kay put 7 stamps. However, 6 more pages contained 5 large stamps each. How many stamps WERE ON THESE PAGES?"	1. 2.
3. "David has 35 stamps, Steve has 62 stamps, and Dan has 14 stamps. The 3 boys are brothers and have a hobby of stamp collecting. Their grandfather also collects stamps and has 105 stamps. DOES HE HAVE MORE STAMPS THAN THE 3 BOYS?"	3. 4.
4. "Carol collects seashells as a hobby. She has been to the beach 5 times to collect 120 seashells. One day her mother told Carol to give 90 shells to six friends. If Carol gives the same number of shells to each person, HOW MANY SHELLS WILL EACH ONE GET?"	5. 6.

KEY:

MANIPULATION:

PART I: MATH TASKS INVOLVED:

1. Principle of subtraction
2. Principle of addition
3. Principle of division
4. Principle of multiplication
5. Principle of addition

PART II:

6. Principle of addition
7. Principle of multiplication
8. Principle of subtraction
9. Principle of division
10. Principle of subtraction

PART III:

11. Regrouping 10's to 1's.
12. Regrouping 1's to 10's.
13. Regrouping 1's and 10's.
14. Regrouping 10's to 1's.
15. Associative Property of addition.
16. Adding Zero.
17. Subtracting zero.
18. Subtracting a number from itself.
19. Multiplying by zero.
20. Distributive property of multiplication over addition.
21. Multiplication and division as inverses.

COMPUTATION:

PART I:

1. Adding zero.
2. Regrouping greater than 10.
3. Compute vertical addition algorithm.
4. Subtracting zero
5. Compute vertical subtraction algorithm.
6. Multiplying by zero.
7. Multiply by powers of ten.
8. Compute vertical multiplication algorithm.
9. A number divided by itself.
10. Division by powers of 10.
11. Compute classical division algorithm.

PART II:

12. Regrouping from 1's to 10's.
13. Place value
14. Regrouping greater than 10.
15. Associative property of multiplication.
16. Distributive property of multiplication over addition.
17. Distributive property of division over addition.

PROBLEM-SOLVING: The 6 tasks which the child must perform to solve each of the problems are:

1. Read the symbols in the problem.
2. Identify the question.
3. Identify the operation.
4. Identify the relevant data.
5. Set up the data in algorithmic form.
6. Compute the algorithm.

The items above are only examples and though they might prove useful at 4th grade or above as an assessment as they are, adjustments would need to be made with other examples at appropriate levels of concept development for younger learners.

Diagnostic Testing and Teaching Through Constructing and Using A Hierarchy of Mathematical Tasks*

Diagnosis may be conducted on 3 levels:

1. general diagnosis
2. analytical diagnosis
3. prescriptive diagnosis

General Diagnosis:

Survey tests (achievement tests) only give information about the learner's general mathematical performance and are not very helpful in identifying specific math competencies and deficiencies! The reason for this is that adequate performance on an achievement test does not necessarily mean that the learner has no mathematical deficiencies or potential problems. Many high-achieving primary-level learners decline in their mathematical abilities and scores year after year until they become mediocre performers or failures. Also, many learners can compute but cannot use their knowledge to solve problems or even conduct their daily activities when mathematics is involved. Why? These learners are often the victims of an early teaching program which allowed learners to substitute memory for understanding. This often happens when learners are encouraged to attempt computational exercises before they have a sufficiently broad base of understanding of the operations concerned. Achievement tests serve few helpful purposes for diagnosis!

Analytical Diagnosis:

These kinds of tests identify specific mathematical competencies and weaknesses of an individual; however, they primarily "categorize" a learner's strengths and weaknesses to classes of mathematical behavior, such as place value, basic addition facts, etc. Most Analytic Diagnostic tests developed commercially operate at this level of diagnosis. The real deficiency in these tests rests with the probability that the learner's specific underlying problems are not identified and analyzed because the learner's diagnosis stops at this "categorical level." This level of diagnosis is only helpful to narrow the search for the more generic and deep-rooted problems. To get at these generic problems the teacher must utilize the Prescriptive Level of Diagnosis.

Prescriptive Diagnosis:

This paper primarily describes this level of diagnosis which utilizes an analysis of the hierarchy of mathematical tasks involved with a specific mathematical competency and deficiency.

Prescriptive Diagnosis is used to determine the nature of a deficiency or the rather exact nature of a learner's learning difficulty. Tests at this level must generally be constructed by the teacher because they are designed for very specific purposes.

Examining a Mathematical Sequence

Before you construct a prescriptive instrument, you must first determine the mathematical skills which need to be examined. For example, if you want to determine the reasons for a learner's inadequate performance on the basic addition facts, you must know the mathematical competencies essential to such performance. Further, you should identify what skills or understandings are not relevant to that task. This requires that you identify the sequence of mathematical learnings implicit in your math program. Such a sequence may be called a hierarchy. A hierarchy is a pictorial representation showing a specific terminal task of a learner, the subordinate mathematical skills, principles and concepts, and their interrelationships. Figure 1 shows such pictorial representations:

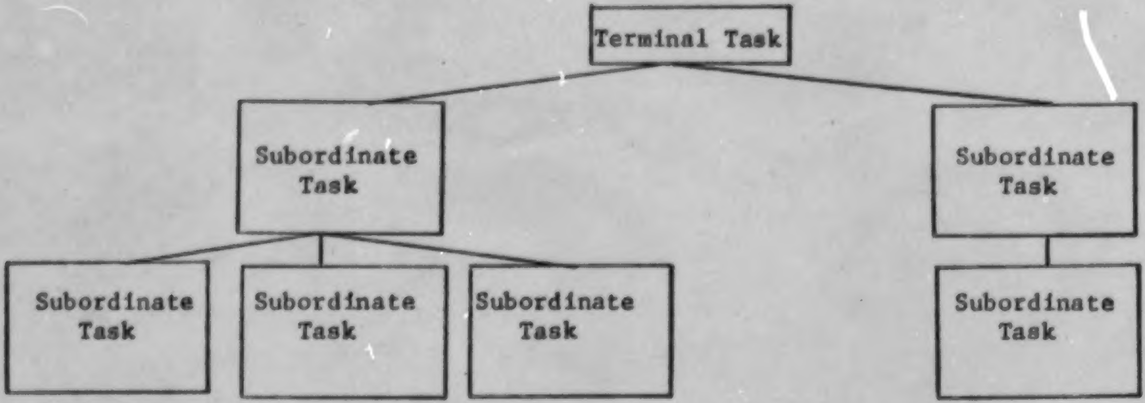


Figure 1

In Figure 1, within each rectangle is the specific mathematical task required to successfully attain the next higher task. The terminal task is shown in the rectangle at the top of the figure. All of the other rectangles, containing skills, principles, and concepts, are pre-requisite (subordinate) to the terminal task. In other words, the subordinate mathematical tasks must be mastered by the learner before he can perform the terminal task. A prescriptive task based upon such a hierarchy then reveals not only a learner's performance on a terminal task but also provides the teacher with information so that he may determine which subordinate behaviors are causing the learner's learning difficulty. To construct such a hierarchy and subsequently a test, the relationship of the subordinate tasks to the terminal task must be explicit and clear. Be certain that the terminal task is well defined so that it provides direction in the construction of the subordinate tasks necessary for its attainment. Look at this terminal goal statement--it precisely and clearly identifies the type of problems learners will be expected to work:

> "Given a set of five written story problems, the learner must correctly recognize those problems involving multiplication, write the relevant data in a vertical multiplication algorithm, and compute the correct answer to each problem."

Note that this goal statement was explicit about the following:

1. Five written story problems will be given.

2. It implies that some of the problems may not involve multiplication and that learners are to work only the problems involving multiplication.

Terminal goal (task) statements should include these guidelines:

1. The conditions of the goal should be clearly and precisely stated.

2. Another person should be able to construct a problem meeting the conditions of the goal by using your goal statement.

3. The type of problem should be clearly specified.

4. The exact behavior of the learner should be described so that a decision can be made regarding his knowledge of the goal.

5. The degree of accuracy should be given. This may be stated as a ratio (4 out of 5 problems), a percentage (90%), or some other means.

Constructing a Hierarchy of Mathematical Tasks

Below is an example of a hierarchy of mathematical tasks from the statement of the terminal task to the pictorialized Hierarchy:

Terminal Task:

"Given a set of 5 written story problems, the learner must correctly recognize those problems involving multiplication, write the relevant data in a vertical multiplication algorithm, and compute the correct answer to each problem."

Sample story problem which would test the attainment of the above task statement:

"John's father has a 640 acre farm. On January 7, he purchased 252 acres. The farm's major crop is oranges. John's father has 28 rows of orange trees. Each row contains 426 trees. How many orange trees are growing?"

This problem satisfies the conditions of the terminal goal--it is written, it contains some data not relevant to the problem, and it can be solved by organizing the numerals into a vertical multiplication algorithm.

Pre-Requisite Tasks:

These must be identified and one can do so by actually solving the story problem and identifying the mathematical skills, concepts, and principles you needed to solve the problem meaningfully. If you were to complete this problem and identify each step you followed, you would also have identified many pre-requisite tasks. For instance, you first had to read the problem, then identify the question, choose an operation, and select the relevant data. At this point, you would be ready to write a multiplication equation. These were the steps you had to follow prior to performing any computation!

After you have the relevant data written in vertical algorithm form, you can begin to compute your answer. Figure 2 below shows the computation process and identifies several of the essential tasks:

426	Operation of multiplication	
x 28	Knowledge of multiplication facts	
48	8 x 6	
160	8 x 20	Distributive property of multiplication over addition
3200	Regrouping	
120	Formation of partial product	
400	Placement of partial product	
8000	Multiplication by multiples of 10	
11928	8 x 400	Associative property of multiplication
	400 x 8	Multiplication by zero
		Vertical addition algorithm
		Associative property of addition
		Commutative Property of multiplication
		Ability to compute vertical algorithm

Figure 2

Now, the tasks involved in Figure 1 are placed in a hierarchical form to show their relationships to the terminal task and to each other. This was done by asking the question:

> "What pre-requisite mathematical tasks must the learner possess to successfully perform this task?"

As you study Figure 3, a completed hierarchy for the above terminal task, you may find one important learning:

> That many sub-ordinate tasks which you, as an adult learner, might assume or take for granted, is not obvious or even known conceptually by younger learners. It is important to realize that such a mechanism as a vertical multiplication algorithm is a composite of many complex and hierarchical skills and concepts!

For clarity of interpretation, Figure 3 hierarchy will be discussed by sequentially numbered explanatory statements:

1. Two tasks were identified under the terminal task when the question was asked about pre-requisites.

Set up a vertical algorithm

Ability to compute the vertical multiplication algorithm

These 2 tasks occur at the same level, since neither is pre-requisite to the other. Since they are the last 2 tasks that learners must be able to perform prior solving a vertical multiplication problem, they appear immediately below the terminal task. Further, once the relevant data is written in a vertical algorithm, learners can perform the necessary computations. Thus these 2 tasks operate at the same time and occur on the same level of the hierarchy.

2. Apply the same question to each of these 2 sub-ordinate tasks and you identify 2 additional sub-ordinate tasks:

   Write multiplication equation

   Placement of partial product

3. The next task of the learner was to demonstrate that he can determine the partial product:

   Determine the partial product

   The learner must be able to do this before he can correctly place the partial product in a vertical algorithm. Hence, "determination of a partial product" is said to be pre-requisite to the placement of the partial product.

4. The same procedure continues until all of the skills, principles, and concepts identified are correctly placed in the hierarchy. For instance, note on Figure 3 the separation of the knowledge of multiplication facts into 2 groups:

   (1) facts whose products are less than 10

   (2) facts whose products are greater than 10

   The first group does not require the place-value skill of regrouping ones to tens. Likewise, the principle for multiplication by one is useful but not essential when computing.

Figure 3 points to the interrelationships which exist among the hierarchies for the four operations on whole numbers. Figure 4 illustrates such a "Composite Hierarchy" for the four operations on whole numbers. A study of Figure 4 as it relates to problem solving should indicate the reason why. Prescriptive Diagnosis by constructing hierarchical task outlines would be helpful to aid the learners who have difficulties with some sub-ordinate task level in some terminal task.

"Hierarchy of Multiplication Tasks"

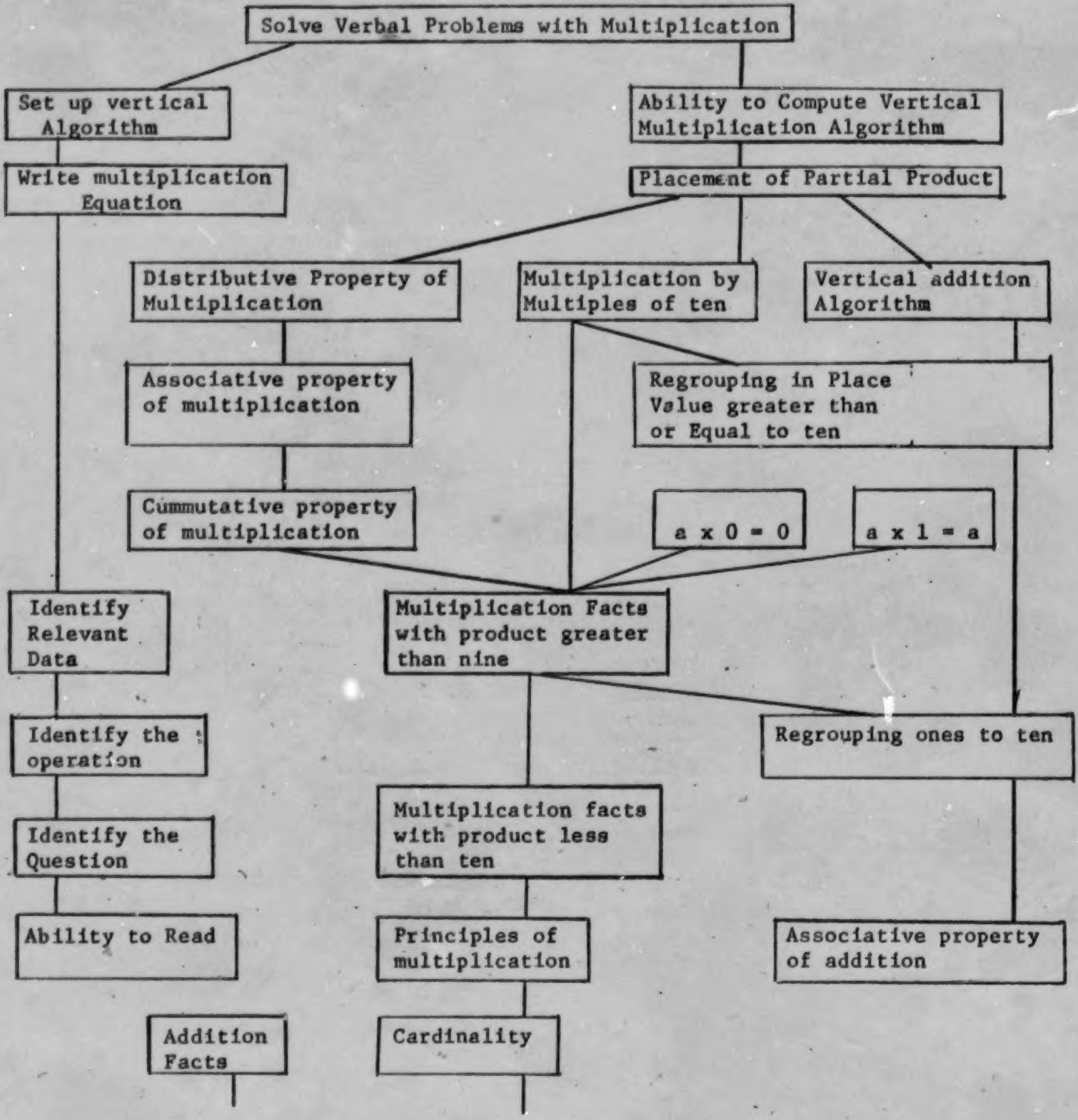

Solve Verbal Problems with Multiplication
Set up vertical Algorithm
Ability to Compute Vertical Multiplication Algorithm
Write multiplication Equation
Placement of Partial Product
Distributive Property of Multiplication
Multiplication by Multiples of ten
Vertical addition Algorithm
Associative property of multiplication
Regrouping in Place Value greater than or Equal to ten
Cummutative property of multiplication
a x 0 = 0
a x 1 = a
Identify Relevant Data
Multiplication Facts with product greater than nine
Identify the operation
Regrouping ones to ten
Identify the Question
Multiplication facts with product less than ten
Ability to Read
Principles of multiplication
Associative property of addition
Addition Facts
Cardinality

Figure 3

Figure 4

Composite Hierarchy of the 4 operations on Whole Numbers

Levels				
28.	Solve Verbal Division Problems			
27.	Set up Classical Division Algorithm	Compute Classical Division Algorithm		
26.	Determine, Place Partial Product	Determine, Place Partial Quotient	Determine, Place Partial Dividend	
25.	Solve Verbal Multiplication Problems	Rounding Off	Apparent	
24.	Set up Vertical Multiplication Algorithm	Compute Vertical Multiplication Algorithm		
23.	Determine, Place Partial Product in Multiplication			
22.	Distributive Property of Multiplication	Multiplication by Powers of Ten	Division by Powers of Ten	Distributive Property of Division
21.	Associative Property of Multiplication			
20.	Commutative Property of Multiplication			
19.	a x 0	a x 1	a/a	a/1
18.	Solve Verbal Addition or Subtraction Problems with "N" Addends	Multiplication Facts	Division Facts	
17.	Set up Vertical Addition and/or Subtraction Algorithm	Compute Vertical Addition Algorithm	Compute Vertical Subtraction Algorithm	
16.	Regrouping Greater than Ten			
15.	Associative Property of Addition	Multiplication & Division as inverses		
14.	Commutative Property of Addition	Principle of Multiplication	Principle of Division	
13.	a + 0	a - a	a - 0	
12.	Addition Facts, Sum Greater than Nine	Subtraction Facts, Minuend Greater than Nine		
11.	Regrouping by Tens and Ones			
10.	Addition Facts, Sum Less than Ten	Subtraction Facts, Minuend Less than Ten		
9.	Addition and Subtraction as Inverses			
8.	Princple of Addition	Principle of Subtraction		
7.	Property of Set Union	Cardinality	Writing Symbols	
6.	Set Union	Counting	Set Separation	
5.	Equivalence	Less Than, Greater Than		
4.	Empty Set	1-1 Correspondence		
3.	Recognition of Sets (Set Properties)			
2.	Grouping by Physical Properties	Discrimination of Items Not in Set		
1.	Recognition of Physical Characteristics (texture, color, shape)			

The composite hierarchy, in Figure 4, shows the pre-requisite tasks for the addition, subtraction, multiplication, and division of whole numbers.

Try writing a terminal goal yourself for any person completing the composite hierarchy shown in Figure 4:

Your terminal goal:

____

____

____

Now, compare your above terminal goal with the following "possible terminal goal":

> "Given ten written or orally presented story problems involving whole numbers, the learner will do the following: (1) identify the operation necessary to solve the problem, (2) identify and write the relevant data in the appropriate algorithm, and (3) compute the answer correctly for 90% of the problems."

There are several tasks which are pre-requisite to all four operations on whole numbers. Such tasks that should be identified at the bottom of the composite hierarchy (Figure 4) are:

1. Recognition of physical characteristics.
2. Grouping of physical properties.
3. Discrimination of items not in a set.
4. Recognition of sets.
5. Empty set.
6. One-to-one correspondence.
7. Equivalence.
8. Less than or greater than.
9. Set Union.
10. Set Separation.
11. Counting.
12. Properties of set union.
13. Cardinality.
14. Writing symbols.

Since these tasks are pre-requisite to the work on any of the four operations on whole numbers, this group may be called a READINESS GROUP. The importance of this READINESS GROUP cannot be overemphasized, for if a learner fails to understand these fundamental concepts and principles, he is almost certain to experience failure in his work in mathematics.

Examine level 8 of the composite hierarchy (Figure 4). Notice that the principles of addition and subtraction occur on this level. Here learners are taught to record in written symbols their actions on physical objects. Addition, as the example below shows is a "joining action":

○○○ ←— ○ ○

"Three circles joined by 2 circles.
There are now 5 circles."

When recording in written form, this action on physical objects appears as 3 + 2 = 5. Likewise, subtraction is interpreted as a "separating action." (NOTE: Subtraction that deals with more than or less than is often interpreted as a "comparison situation" rather than a "separating action.")

If a learner can demonstrate the operations of addition and subtraction (1) What is the next task which the learner should try? (2) Why is "this" task taught at this level?

The next task to be learned is that the operations of addition and subtraction are "inverses." Understanding of the two operations is greatly facilitated by emphasizing their inverse relationships. In fact, Piaget believes that if subtraction is stressed as a separate operation, it should still be taught at the same time as addition and the same problems used so that "reversibility of the concept" is seen. If reversibility is not seen, addition is not fully learned, and it may be only a rote or memory process for the learner.

Understanding of the inverse relationships between addition and subtraction allows a learner to progress to 2 more tasks. These tasks occur on level 10 (Figure 4):

1. Addition facts, sum less than 10 versus subtraction facts, minuend less than 10.

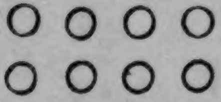
2. Addition facts, sum greater than 9 versus subtraction facts, minuend greater than 10.

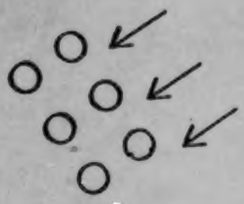
The tasks in 1. above may be illustrated by such equations as:

3 + 4 = 7   7 - 4 = 3   7 = 3 + 4   3 = 7 - 4
4 + 3 = 7   7 - 3 = 4   7 = 4 + 3   4 = 7 - 3

These equations thoroughly show the relationship between addition and subtraction. Learners should be able to manipulate physical objects, write equations, and verbally explain the inverse relationship illustrated by the above equations.

Why in the 2 tasks above were the facts of addition and subtraction separated into 2 groups? Because some addition facts require learners to form groups of ten and ones to determine sums--for example, meaningfully recording the addition fact 8 + 6 requires "regrouping" from ones to tens:





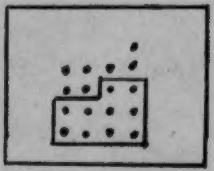
8 ones + 6 ones = 8 ones + 2 ones + 4 ones

= 1 group of ten + 4 ones

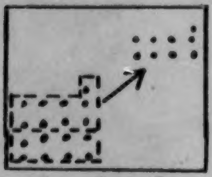
= 10 + 4

= 14

Also, when computing the answer to some subtraction facts, learners must "regroup" from the tens place to the ones.



10 + 7 = 17



17 - 9 = 8

17 - 9 = (1 group of ten + 7 ones) - 9 ones

= 17 ones - 9 ones

= 8 ones

= 8

The regrouping in the tens and ones places serves as an introduction or a foundation to regrouping in any place-value position. Learners may generalize from regrouping with ones and tens so that they apply the regrouping process to regrouping with hundreds, thousands, etc. This ability to apply the generalized regrouping process is pre-requisite to the successful application of the addition algorithm, subtraction algorithm, and other higher order tasks. Note, however, on Figure 4. that regrouping greater than 10 is not needed until immediately prior to the addition and subtraction algorithms, so it occurs on level 16. All of the tasks below this level can be developed independently of the ability to regroup in place-value positions greater than 10. By showing the algorithms for addition and subtraction on the same level should facilitate the learning of both algorithms.

The principle of multiplication and the principle of division occur immediately after the development of the addition and subtraction facts (Figure 4, levels 14 & 15).

The reasons for placing the princples of multiplication and division on similar levels are very similar to the reasons given for addition and subtraction--learners have the knowledge to pursue the study of either. Neither principle is pre-requisite to the other. Also, their inverse relationship must be understood if either is to be understood. Again, Piaget believes the two operations should be taught simultaneously to aid the learners' development of "reversibility."

Placing of multiplication and division principles at this level of the hierarchy (Figure 4, levels 14 and 15), indicates that learners are "ready" to develop the principle of multiplication--they can interpret multiplication as repeated addition or the union of equivalent sets. For example, 6 - 3 means "six sets of 3 objects." By using physical objects learners would form 6 groups of 3 in each group:

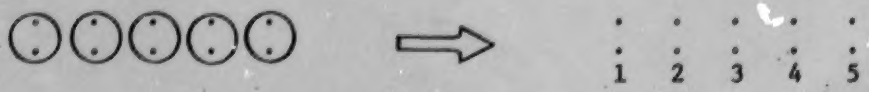
. . .   .

3 + 3 + 3 + 3 + 3 + 3

$6 \times 3 = 18$

Also, Piaget has found that learners develop the ability to perform the operation of logical multiplication at approximately the same time as the ability to perform logical addition.

Similarly, division may be interpreted as repeated subtraction or the separation of a set into equivalent subsets. Consider 10 ÷ 2. By using objects learners would form groups of 2 objects:



$10 \div 2 = 5$

This may be verbalized as "10 divided into sets of 2 makes how many sets?"

Because multiplication and division are inverse operations, several tasks occur on the same level. These tasks listed below should be taught so that the study of one task will facilitate the learning of the inverse task:

1. Multiplication facts versus Division facts.
2. Multiplication by one versus division by one.
3. Multiplication by powers of ten versus division by powers of 10.
4. Distributive property of multiplication versus distributive property of division.

The whole intent of the "hierarchy" examples and efforts is to encourage learners to "generalize from their experiences" with various facts to form principles. By doing so with the multiplication and division facts, these principles would be formed:

1. Multiplication by zero.
2. Multiplication by one.
3. Division by one.
4. Division of a number by itself.

In view of such "generalizations" it makes it pointless for children to "drill" on the zero and ones facts in any of the 4 operations.

After study of the composite hierarchy, it is hoped that teachers can see why learners have so many difficulties with multiplication and division algorithms. They are too often asked to successfully perform tasks without the ability to perform all the pre-requisite tasks. Working with such hierarchies will vastly increase the teacher's skill in both Diagnosis and Individualization.

The Changing View of Learning*

To consider a change in the elementary school arithmetic program is to consider an extremely complex problem. In addition to the important attention given the content of the program and its logical representation, children must be considered. The interaction of known and unknown attributes of children and their learning comprises an amorphous system of possibilities. While a great deal is known of children's developmental characteristics at the various elementary school ages and there is a consensus regarding certain features of children's learning, a vast area of uncertainty accompanies the framing of a program to fit the children. It is clear that the available research on human learning or on curriculum organization has not produced as yet a completely detailed and trustworthy model for elementary mathematics curriculum formulation. Hence, curriculum decisions in this area of consideration are often based upon intuition and the insight which derives from practical experience.

Similarly, teaching practice in today's elementary school classrooms reflects an uncertainty about the manner of learning arithmetic. In some classrooms one finds teachers as transmitters of knowledge, much in the manner of the early scribes and priests who collected number concepts and arranged them for transmission to apprentices. Pythagoras, for example, entered the priesthood in Egypt to learn the closely guarded secrets of the caste and later returned to Greece to establish classes of his own. The members of his classes were expected to memorize the concepts which he had acquired in order that they, in turn, could transmit them in due time. Some teachers today collect concepts and pass them along year by year to the children in their classes, mainly through the devices of extended practice and memorization.

Other teachers use questions as the medium of instruction and, at least to some extent, support what may be called the Socratic method. The questions and answers are designed to produce an integrated body of mathematical knowledge, but the interaction between most children and fundamental mathematical ideas is commonly sacrificed.

And then there are the teachers who resemble the schoolmaster described by Dickens and who follow what may be called the "squeers method." This method consists mainly in the use of arithmetic as a device for creating mental discomfort among children. The amenability of the subject to increasing abstraction and complexity is exploited to the distaste of most sensitive youngsters. In these cases, as well as in many others for which examples could be drawn, there is a learning theory, albeit an individualized teory. Teaching proceeds according to the beliefs learning, and, to a large extent, the preparations of curricula are affected by the views of those directly associated with the preparation.

As indicated in the historical overview of curriculum change, some major trends in school arithmetic were dictated by prevailing views of learning. Certainly the mental discipline curriculum in the last part of the nineteenth century is an example of a program in concert with an accepted view of the nature of the nature of learning. Learning was viewed as a direct product of exercise of the mind. The mind was subjected to exercise just as the muscles were subjected to exercise by the weight lifter. As development of muscles proceeded, more weights were added, and as development of the mind proceeded, more arducous and difficult mental tasks were added. The textbooks which gave form to the program of this era included nothing but isolated and disconnected rules. They appealed to memory almost exclusively and neatly ignored the existence of reasoning powers in the mind of the learner.

The era of connectionism was also directly represented in the arithmetic program. In 1922, E. L. Thorndike had published his classic treatise on arithmetic learning entreatise on arithmetic learning entitled The Psychology of Arithmetic. While his "laws of learning" were often misunderstood and misrepresented, they nonetheless

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*LLOYD SCOTT: Trends in Elementry School Math, Ran McNally Co., 1966, pp. 9-16.

contributed to a practice of separating arithmetic into bits of learning. The appropriate teaching adjustment to the theory and to the dissected program was drill on the segments, facts and manipulations until the proper connections were established. Improper connections were to be avoided at all cost. In the final analysis, the program did produce computational skill, even though the application of the skill to problem solving was seldom made either within or without the school.

What may be called the meaning theory came into prominence in about 1935. This concept of learning arithmetic derived from the field theories of learning based upon Gestalt psychology and was diametrically opposed to the dissections which characterized the program aligned with connectionism. Learning was viewed as deriving from analysis of problems rather than from recalling information. The relatedness of the parts of an entire context was the center of the learning experience, rather than the isolated bits of which it was composed. A challenging problem was viewed as relatively unstructured and learning was viewed as the pursuit of some orderly structure through which the problem could be managed. When the relationship of the parts to the whole was rendered clear, then insight was gained and learning had been achieved.

In general, the meaning theory continues to provide a basis for arithmetic program development. There remain open questions concerning the means through which insight is best achieved, and controversy probably will rage for some time to come. Thus, within the broad limits of the meaning theory there is room for many individual learning theories, each centering in a different point of view. It will be difficult always to know which "theory" is the absolute best in terms of societal goals and the characteristics of the learners, and it is often difficult to stop the most influential approach at a given moment.

However, within the current relevant literature some centers of curriculum influence may be noted. While these influences do not alter substantially the qualitative basis for detailed curriculum decisions, they seem to provide some common ground upon which general decisions may rest.

Notable among the current efforts which have been closely associated with modern curriculum development is the work of Professor Jean Piaget of Switzerland. Professor Piaget is perhaps best known as a developmental psychologist and his work with children has spanned a period of more than forty years. In his recent work, however, his interest in the intellectual and perceptual development of children has centered upon the relationship between thought activities and systems of logic and upon the growth of logical concepts in the individual. Of course, in this part of his activity he plays the role of the logician. Regardless of the particular classical identification of his work, curriculum projects are being increasingly affected by his theory and reported "experimental" results. At the present time, the most influential aspect of his studies upon new arithmetic curricula would seem to be the age-normative basis for intellectual growth which he has defined. In particular, he holds that a school child's ability to conceptualize may be seen to proceed through three rather distinct periods. The first may be designated the pre-operational period. This phase apparently continues from birth to about six years of age for most children. The pre-operational period is followed by the period of concrete operations which lasts until the child is about eleven years old. The third and last period defined by Piaget is designated the period of formal operations and this period extends through adulthood. While the transition points between periods have not been precisely fixed either by Piaget or by others who interpret or replicate his work, the ages of six and eleven may be regarded as rough guidelines.

The pre-operational period is characterized by a transition from the realtively random sensory-motor reactions of the infant to the pre-operational representations of the five-year-old. In general, this stage may be viewed as the stage of heavy reliance upon the senses and the gradual movement toward some symbolic representations for the sense impingements. There is little evidence of the development of a rational or structured conceptual framework.

The period of concrete operations is reserved for such organization of concrete experiences. Concepts are associated into cognitive structures called groupings, and these groupings serve as organization centers for the assimilation of the expanding experiences with the natural world. In this period the child is not able to deal effectively with ideas beyond his range of experience or beyond the reach of his senses. Objects serve to direct his thought, and interaction between the child and environmental objects provides the substance for his conceptual growth.

Piaget describes the period of formal operations as that period when the individual is capable of intellectual activity without the presence of objects. He is able to formulate ideas independent of corresponding sense encounters and he can perceive and analyze relationships among ideas. The learner in this period is attentive to logical development and interested in the orderly arrangement of facts and ideas. He finds it possible to extrapolate and operate beyond the immediately present ideas and he is in a position to extend the organization of knowledge.

Until very recently, there has been no substantial regard for the developmental stages defined by Piaget in the design of elementary school mathematics curricula. Today, however, there is a growing belief that human learning does proceed through some sequential defined stages and that these stages may be characterized by certain mental operations. The sequence of stages may well be invariant, as Piaget believes, though the rate of progress through the stages may be conditioned by certain factors under the control of the school. Furthermore, there is an increasing regard for a kind of analysis of thought activities based upon logical patterns. Piaget has promoted the belief that learning may be studied and classified into systems of operations; then these systems may be analyzed according to certain properties. Among the properties named by Piaget are: (1) composition--operations or suboperations may be combined into a new unit, (2) reversibility--combined units may be returned to original form and (3) associativity--the units may be grouped together in various ways, each with the same result. Such an approach brings a new dimension to the analysis of human thought. Without anticipating its ultimate worth in this important area of investigation, a new approach is somewhat akin to a breath of fresh air.

Another psychologist of influence in contemporary curriculum reform is Professor Jerome Bruner. In particular, his book entitled The Process of Education, which is a written summary of a major curriculum forum, has had a substantial impact upon program planning. Perhaps because the timing of publication of the volume was propitious, his treatment of the use of the structure of disciplines as the central notion in curriculum design was widely accepted and actively supported by major curriculum projects, especially those dealing with mathematics and science. While there were several operating curriculum groups involved in the formulation of programs which emphasized the unifying ideas in various subject matter areas, Bruner's convincing presentation of this point of view became a rallying point for these projects and they were soon joined by other groups impressed by the "structure" philosophy. Furthermore, the motivating aspects of inductive learning were stressed by Bruner, among others, and, despite an apparent conflict with the deductive nature of mathematics, induction and discovery became popular themes for program design. Thus, the selection and organization of program content and the approach to the material were conditioned by the perceived nature of children's learning.

The work of Z. P. Dienes has had some major effect upon the elementary school arithmetic program in transition, but the effect has been felt principally outside the United States. Only recently has his work drawn the attention of curriculum theorists in this country. While Professor Dienes may be best known for his development of mathematics teaching materials, his theoretical orientation has been effectively compatible with the Piaget and Bruner positions. Many designers of elementary mathematics curricula sought program improvement through a theoretical, abstract approach to the content of the mathematics. Dienes seems to follow the observations of Piaget, at least to the extent that the youngster in elementary school is seen to be at the concrete operations level of development. Dienes' development of materials was thus a consequence of his belief that concrete objects were necessary in mathematics learning during the first six or seven years of schooling. He demonstrated that there were different tendencies in thinking associated with motivation. He distinguished constructive thinking from analytic thinking and stressed the importance of constructivity, a process of synthesis of ideas through the manipulation of concrete materials. He saw the use of concrete objects in three sequentially ordered levels of experience. At the first level, the materials are used by children in an unstructured, undirected manner. According to Dienes, this is a necessary preliminary step before the materials can be organized or manipulated in an organized way. At the second level, the child's actions with the materials become directed toward some end. The materials are ordered and classified and symbolic representations, principally words, begin to be used in the production of ideas. At the third level, relationships and structural patterns are perceived by the child and insights are developed and strengthened. In all these levels, the interactions between children and materials are the crucial components of concept development. In essence, Dienes gives strong support to the view that constructive thinking must precede analytic thinking (Piaget's period of formal operations); otherwise there is nothing to analyze.

While the contributions of Piaget, Bruner and Dienes do not offer a unified or comprehensive theory of learning, they are illustrative and commonly cited bases for arithmetic curriculum decisions. It is interesting that there are apparent direct conflicts in the approaches of these theorists. For example, it would seem that Bruner would be satisfied if mention of learning readiness disappeared from the literature. The learning model of Piaget, on the other hand, is directly tied to the existence of readiness in the learner.

At the risk of oversimplifying the diverse psychological theories related to elementary school mathematics, the following detailed themes related to various views of human learning appeared to have received major attention:

1. The structure of mathematics should be stressed at all levels. Topics and relationships of endurance should be given concentrated attention.

2. Children are capable of learning more abstract and more complex concepts when the relationships between concepts is stressed.

3. Existing elementary arithmetic programs may be severely condensed because children are capable of learning concepts at much earlier ages than formerly thought.

4. Any concept may be taught a child of any age in some intellectually honest manner, if one is able to find the proper language for expressing the concept.

5. The inductive approach or the discovery method is logically productive and should enhance learning and retention.

6. The major objective of a program is the development of independent and creative thinking processes.

7. Human learning seems to pass through the stages of pre-operations, concrete operations and formal operations.

8. Growth of understanding is dependent upon concept exploration through challenging apparatus and concrete materials and cannot be restricted to mere symbolic manipulations.

9. Teaching mathematical skills is regarded as a tidying-up of concepts developed through discovery rather than as a step-by-step process for memorization.

10. Practical application of isolated concepts or systems of concepts, particularly those applications drawn from the natural sciences, are valuable to reinforcement and retention.

E.E. Biggs

> True or living learning originates from the child and indeed directly from some of his stronger interests and drives.
>
> --Nathan Isaacs, Psychology, Piaget and Practice

In this chapter we shall review the work of Piaget and others concerning the learning of mathematical concepts by young children. We shall then attempt to decide what influence their findings ought to have on our teaching of the early stages of mathematics.

Let us first turn to D.O. Hebb, of McGill University, for an unequivocal statement about the requirements of intellectual growth. 'There are two determinants of intellectual growth: a completely necessary innate potential and a completely necessary stimulating environment.'[1] If we accept this statement (and it is supported by the weight of prolonged research), our attention as teachers is directed once more to the importance of the environment. As we shall see, Piaget in his experimental methods sets us a pattern for use in planning our learning situations. In his preface to THE CHILD'S CONCEPTION OF NUMBER he writes, 'Conversation with the child is more reliable and more fruitful when it is related to experience with adequate material, and when the child, instead of thinking in the void, is talking about actions he has just performed.'[2]

Piaget set himself the task of finding out, as accurately as possible, how the principles of conservation and of reversibility, as applied to numbers and to spatial thinking, develop in the minds of young children. These two principles are fundamental to all mathematical (and logical) thinking. For example, in the field of numbers, conservation means that the number of objects in a group remains the same however the objects are arranged (in a heap, in a long line, etc.). Or if we are considering quantities, a quantity of lemonade remains the same if it is poured, for example, from a shallow dish into a tall, narrow glass. The understanding of reversibility involves a realisation that reversing an action would result in a return to the original state of affairs.[3] Piaget devised a variety of tests and tried these out on children of pre-school and primary school age.

Although the ages are always quoted in the records, it is the stages of learning to which Piaget draws attention. He was the first to discover that the formation of a concept takes far longer than had been supposed. His findings, and the stages of learning he postulates, are most easily understood in reference to one of his well known simple experiments.

This experiment was the third of a series concerned with one-to-one correspondence. In the test, the child was shown a set of seven egg-cups in a row and also a group

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* From Mathematics in Primary Schools, Curriculum Bulletin No. 1, Schools Council, London, England, 1965, pp. 5-9.

[1] The Organisation of Behaviour, p. 302. New York, 1949.

[2] The Child's Conception of Number, 1952. La Genese du Nombre chez l 'Enfant, Geneva, 1941.

[3] Not all operations are immediately reversible. When we let a bath of water drain away we cannot get the same water back again.

of eggs (containing more than seven eggs). He was asked to take just enough eggs for the egg-cups. In Stage I (normally between the ages of four and five) one child made a row of the same length but containing too many eggs. He was then asked to put the eggs into the egg-cups to check his answer and was surprised to find there were too many eggs. The extra eggs were removed and the child agreed that now there were the same number of eggs and of egg-cups. But when the seven eggs were taken out and put in a heap in front of the egg-cups the child said that there were more egg-cups than eggs. It was clear from the similar responses of many children that, at this stage, children are not capable, by themselves, of making the one-to-one correspondence and that they would not have discovered it if the relations between the egg and its cup had not forced them to do so. As for the equivalence of the two sets (eggs and egg-cups), the child's answers were based entirely on a visual (perceptual) comparison of the length of the rows, even when one-to-one correspondence had been established by the nature of the materials.

In Stage II (normally between the ages of five and six) the child, of his own volition, took seven eggs to correspond to seven egg-cups and put the eggs in the cups. When the eggs were removed and spaced further apart, the child said there were more eggs than egg-cups. When asked if there would be the right number of eggs to put one egg back in each cup he did not know. Here the child created one-to-one correspondence for himself, but he no longer recognised the equivalence of the two sets once the configuration was changed.

In Stage III (normally between the ages of five-and-a-half and six-and-a-half) the child achieves 'operational correspondence and lasting equivalence.' Even when the eggs were spread out the child maintained that the number of eggs and egg-cups were the same, 'because they all go into the egg-cups.' It is only at this stage that reversibility is understood; that is, the child can reverse the thinking process and put the eggs back into the egg-cups in imagination. So the child has achieved full operational control of the concept.

Piaget has thus drawn attention to the fact that there are variations in kinds of thinking according to chronological or mental age. Of these kinds of thinking, Dr. E.A. Lunzer in THE WORK OF PIAGET AND ITS RELEVANCE FOR THE TEACHER[4] writes: 'The ability to carry out actions in our heads, actions of ordering and reordering, and of appreciating equivalences . . . is what Piaget calls operational reasoning. Such reasoning always involves systematic equivalences; for example, doing something (like adding) and then doing the opposite (like subtracting the same number) brings us back to where we began.'

. . . .

The importance Piaget placed on the use of the spoken word in his experience with children raises the issue of the place of language in children's learning. In a carefully controlled piece of research work with a limited number of English children between the ages of 5.3 and 5.9 Mrs. J. Tough[5] investigated the contributions made by relevant experience and language to the formation of a number concept by five-year-old children.

Of the four groups considered, the group with whom the appropriate language was introduced at the same time as the experience made significant progress in the learning of the concept under consideration. Half the children in the group having experience without appropriate language had made some progress but in two control groups very little progress had been made. In Mrs. Tough's own words: 'Of the three situations explored, the basic concept is more efficiently formed when language and concrete experience appear together. For the schools this means the provision of good concrete experience together with active teacher-participation and the stimulation of a discussion.' Although this experiment involved a relatively small number of children the results are decisive and merit much further research. We are therefore left with the following three questions. Are we sufficiently aware of the concepts basic to later mathematical understanding? Do we provide the right kind of concrete experience during the early years? Do we provide at the same time language which helps efficient organisation of the experience?

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[4] Unpublished manuscript. Manchester University.

[5] A study of the contributions made by relevant experience and language to the formation of a number concept of five-year-old children. Unpublished thesis, 1963.

This investigation shows the significance which one aspect of Piaget's work can have for teachers. What significance has Piaget's work in general for us as teachers? Once more let us turn to the experts in this field for more specific guidance. Dr. Lunzer[6] writes: 'The principal field for the application of Piaget's methods and results in educational practice is that of curriculum and method.' Of children's understanding of number he writes that such understanding cannot be taught nor does it come by itself, independently of experience. It is the fact that number is an integral part of our daily lives which ensures that children do come to this understanding.' He can provide the kind of experience which will assist the child to move from intuitive to operational thinking. 'This (operational thinking) means activity, but activity in the sense of mental activity.' Not only must the experience provided be of the right sort; it must occur in a context in which the child is interested in the result. From this we can conclude (1) that we can promote understanding of mathematical concepts by providing experience of the right kind, (2) that the experience or activity provided is not an end in itself although it is planned to arouse interest; it is devised with the declared purpose of stimulating spontaneous discovery by the child.

Now if we want to plan the right kind of experience we must know (1) the mathematical concepts we want the children to learn, and (2) how children learn.

We shall turn our attention to the first problem in subsequent chapters. Piaget's methods help us with the second for he based his conclusions on his observation of young children. It is almost impossible for us as adults either to remember how we learnt mathematical concepts or to imagine ourselves ignorant of them (imperfect though our knowledge may be). Therefore it is essential, if we want to discover how children learn, to plan learning situations for them and to observe what they do. It was by observing their actions, and by questioning them and listening to their answers, that Piaget discovered how the children were thinking and reasoning and learning.

Now let us see what a research worker in a different field has to say about Piaget's work. Dr. Z.P. Dienes relates Piaget's three stages of forming concepts to different types of learning situations. He calls the first the preliminary or play stage and writes: 'In order to make possible, freedom to experiment is necessary.'[7]

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[6] op. cit.

[7] Building up Mathematics by Z.P. Dienes, p. 39. Hutchinson, 1960.

This stage is marked by an undirected and seemingly purposeless activity. The second stage is more directed and purposeful but there is no clear realisation of what is being sought. At this stage a certain degree of structured activity is desirable but, because children think in different ways, Dr. Dienes recommends the provision of a number of experiences of varying structure, all leading to the same concept. The third stage must provide the practice necessary for fixing the concept. He therefore refers to: (a) preliminary games, (b) structural games, and (c) practice games, but adds, 'Clearly a practice game for one concept can act as a preliminary game for a later concept. It is important, however, not to use practice games as preliminary games for the same concept, a common error in infants' schools.'

. . . .

With discovery methods in mind and encouraged by Piaget's experiments, let us create a dynamic definition for the learning of mathematics. To be brief we might content ourselves with: 'Mathematics is a discovery of relationships.' But if we (or the children we teach) have discovered such a relationship for ourselves we want to communicate the exciting discovery to others. We may first describe the discovery in words to a friend or teacher. Subsequently we may find a more effective way of expressing the relationship; for example, in numbers (as in arithmetic), in letters (as in algebra), by a diagram (as in geometry) or by a graph. Therefore a more comprehensive definition would be, 'Mathematics is a discovery of relationships and the expression of the relationship in symbolic (or abstract) form.' This is no static definition but implies action on the part of the learner, of whatever age and whatever ability. It is the fact that mathematical relationships can be discovered and communicated in such a variety of ways that puts mathematics within reach of children and adults of all abilities. In the following chapters we shall show how young children can learn mathematics by their own efforts and how we can plan the learning situations which will effect this. For this purpose we may summarise the conclusions derived from the research to which reference has been made in this chapter as follows:

1. Children learn mathematical concepts more slowly than we realised. They learn by their own activities.

2. Although children think and reason in different ways they all pass through certain stages depending on their chronological and mental ages and their experience.

3. We can accelerate their learning by providing suitable experiences, particularly if we introduce the appropriate language simultaneously.

4. Practice is necessary to fix a concept once it has been understood, therefore practice should follow, and not precede, discovery.

Drastic changes in the mathematics curriculum would not be possible if we were to retain the traditional classroom procedures and atmosphere at the same time. In fact, we hope that teachers will endeavour to change "a teaching situation" into "a learning situation." It must be emphasized that, with the kind of approach suggested here, far less "whole class teaching" will take place. Much of the work will be done by children working in small groups, or even individually. These groups can be formed by the teacher, or she may allow the children to form themselves into groups. They will work together quite happily, especially if the work is not spoiled for them by the creation of a reward-punishment system. Children are essentially interested in finding out new things about their world, and we do not have to spoil this interest by introducing compulsions or rewards for work well done. A smile from the teacher, or a pat on the back, is quite sufficient reward for a task well completed.

If we work in this way, the children will be encouraged to learn mathematics for its own sake, and not in order to excel or outdo their classmates in competition. Groups will form, will change in composition, and re-form, as some children learn more quickly than others. There will be a place for individual work, too, and there will be times when it will be more profitable for the while class to work together. A case in point occurs in the study of sets. Sets are probably most satisfactorily introduced by considering the children in the class as possible members of any sets. In other words the universal set can, in the first place, be defined as the set of all the children in the class. At a later stage, too, equivalence games are most easily played by using the children themselves. When the question is asked, "Are there more chairs than children, or more children than chairs?" the children will readily find out, if they are allowed to do so, by trying to sit on the chairs. If all the children are able to sit down, one to a seat, and there are some empty chairs, then, of course, there are more chairs than children--and nobody needs to know the exact numbers of children or chairs. In these situations the joint experience of the class, or at any rate of a large part of the class, will be beneficial. So no hard and fast rules can be laid down as to whether individual work, small group work, or whole class work is the best way to handle a situation. It must, in the last resort, be left to the teacher to choose what he feels is the best approach in the situation in which he finds himself. Very frequently, when a new aspect is to be introduced, the class will be best taken as a whole. This activity may not last long, for it may be found that progress has been so varied that the next stage must be taken in groups--and this may be so even before one lesson has been completed.

An important part of learning takes place by discussions among the children. To illustrate this let us take the case of a logical game, with a Venn Diagram being completed on the floor. If a child puts a piece in the wrong place, it is much less useful for the teacher to interfere than for another child to correct it. The two children can argue the matter out on the same level, and the child who thinks the piece has been wrongly placed will usually argue quite forcibly, while the other child argues the point back again. The rules of the game are quite simple enough for the truth to prevail eventually in such an argument. This is very good training, as it is very much better to encourage children to appeal to the truth in a situation than to appeal to the authority of some person who is a purveryor of truth, such as a teacher.

When children are encouraged to discuss not only what they are doing but also what they believe they have discovered, quite naturally there will be a certain amount of noise in the classroom. However, there is no need to allow excessive noise to prevent learning from taking place or to interrupt the activities of other classes. The teacher must insist that necessary noise be limited. But, it is amazing how much noise a child is able to support, while yet doing a considerable amount of delicate thinking. It is usually the teacher who is "driven mad" by the excessive amount of noise, and not the

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*From Learning Logic, Logical Games, by Z. P. Dienes and E. W. Golding, pp. 10-14. New York, Herder and Herder, 1966.

children. On the other hand, just as the teacher must get used to a more noisy situation, the children must learn to consider others. We have found that, with a little "give and take," this problem is usually solved satisfactorily.

If the children learn best by activity methods, and if discussion will help such learning, the teacher must adjust to the new situation; while, if children are to learn in the usual school situation with other classes nearby, they must limit the volume of noise that they create.

In the implementation of a scheme of learning such as described in this handbook, there will be a great amount of concrete material to be handled by the children and by the teacher. Taken in conjunction with group and individual work, this calls for a considerable amount of organization; if the activities and the materials are not properly organized there will be chaos, loss of time, and poor learning conditions. One way of making sure that every child knows exactly what to start with at the beginning of a lesson is to put a diagram on the blackboard, with the names of children or groups beside each section of the diagram. Sketches are sometimes necessary because children cannot yet read instructions: thus we might use three interlocking circles for a Venn Diagram, or a quick sketch of some blacks to show an exchanging game. The materials, too, must have a definite place in the classroom, or in the corridor, where the children are able to reach them, so that, before the mathematics lesson, the teacher can simply ask certain responsible children to collect materials, put them where they will be needed, and distribute cards. At the end of the lesson, children may again be asked to check materials, pack and replace them tidily in the cupboards.

Once such an organization has been established there seems to be no trouble in getting the class started, but it certainly does need organizing; it will not happen by itself. Except at such times as when a new aspect is to be introduced, teachers will be well advised to "stagger" the changeover periods of the groups or of the individual children so that the beginning of a lesson sees most children continuing an activity of which they have some knowledge, while only a few must be introduced to something new. This, too, calls for considerable ability as an organizer.

It is not possible for a traditionally trained teacher to pass over to this kind of mathematics without a certain amount of heartsearching, and a consequent change in attitude. For example, the idea that truth is the authority, and not the teacher, is something rather difficult for some teachers to accept. Children themselves are accustomed simply to "ask the teacher," It is very tempting to interfere with children when they are making mistakes, and tell them how to do things when they don't know. It is quite difficult to stand by and watch a child fumbling away, not being able to solve a problem, when all the teacher would have to do would be to say, "Look, put it over there," and it would be done. This, of course, would rob the child of the benefit arising from that particular learning situation in which he is expected to discover for himself what the solution is. By solving it for himself, he has the opportunity of fixing the solution in his mind very much more clearly and permanently than if the teacher merely tells him what to do.

In addition teachers should try to remember that the ways in which they think are not necessarily the ways in which the children think. In fact, child-thinking is very different from adult-thinking, and even different children seem to think problems out in different ways. There is no single way to solve a problem. Very often a child, given the opportunity, will suggest an avenue of attack for a problem which is not the avenue that the teacher himself would have chosen--in fact, the line of attack might seem to him to be entirely wrong. The best educational approach in such a case would have the teacher avoid saying, "That is wrong: do it this way," and have him join with the child in an investigation of what is entailed in the child's suggestion. A discussion, or a joing piece of discovery, could then follow, with the child's approach examined for what it is worth. If it is a good one, and if the child is intelligent enough to follow it up, he may convince the teacher, If this does not happen, and the child goes on fumbling and finding that the method is not particularly successful, them it will be time for the teacher to suggest that some other line of attack might be advisable.

It should not be thought, because it is suggested that children should not be unduly interfered with, that this implies that they should always be left entirely to their own devices. Occasional helpful suggestions from the teacher are a very necessary part of the learning process, but these should never take the form of commands. A child's mistake should not be pointed out to him in so many words, even though it be apparent to the child. He must realise his result is absurd, and it will then be borne in on him that his approach was incorrect. It is very much better to discover one's own mistakes than to be told about them by someone else, and in the process it is found that more is learned about the structure of the problem, for he has not had any personal active experience of handling it.

MANAGING MATHEMATICS IN THE EARLY EDUCATION OF CHILDREN

G. Schlinsog*

Our concepts of early childhood education are being altered rapidly by those active in the study of child growth and development. The established fact that a major portion of the shaping of the intellect takes place before a child reaches the age of seven (1) is startling, and it imposes tremendous responsibility on those interested in teaching young children. Equally crucial and influential as it relates to the mathematics education of young children has been the enunciation of three distinguishable stages (2) along the continuum of quantitative cognition. The early years are crucial in the development of such aspects as language ability, personality, intelligence, and the ability to think. The growth of a child's thought processes depends upon his having something to think about and upon his opportunities to have appropriate new experiences. What he is able to assimilate from those experiences depends partly on the thinking processes he has already developed. Patterns of abstract thought emerge slowly as the child encounters new ideas, reacts to them, and incorporates them into his own intellectual behavior.

The 1960's saw the beginning of a number of projects under the general heading of compensatory education. These projects have focused attention on the lasting effects of deprivation during a child's early years. Experimentation and research are providing insights into the possibilities of arresting and even reversing the effects of cultural, social, or experiential deprivation. Carefully planned kindergarten and preschool programs are being used to encourage language development, to stimulate the growth of cognitive abilities, and to guide the formation ofselected concepts in the minds of young children.(3)

All of these events have contributed to the accumulating evidence that the environment and experiences of the infant and young child are more crucial or decisive in developing his capacity to function as an adult than all his later years of schooling . Altogether, these changing emphases would seem to point to new types of kindergarten and preschool programs in mathematics. To the conventional concern for cognitive development must be added a refurbished dimension, a realistic concern for the affective development; that is, the establishment of values, interests, appreciations, and attitudes. The latter deals with emotions; that is, feelings, awareness, individual learning styles, and pupil acceptance of the learning environment.

Present evidence points to the emergence of new early childhood programs having language development as their core. (4) Within these programs, there is a definite place for mathematics as a prerequisite for perceiving, discussing, reasining, and accepting the quantitative aspects of life. There are key mathematical concepts, important for later learning, which have significance and interest for children five, six and seven years form of play, and the fascination frequently continues until formal written work and pressure for conformity begins to dull the child's interest.

APPROPRIATE MATHEMATICAL CONCEPTS FOR YOUNG CHILDREN

It is appropriate at this point to raise a question as to what mathematical concepts are suitable for development at the early age of four to seven years. Although it might decrease the readability of this article, a list of possible concepts is suggested below. To conserve space they are presented in outline form.

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Concepts and skills in early mathematical development:

- I. Prenumber concepts
  - A. Longest = shortest
  - B. Tallest = shortest
  - C. Biggest = smallest
  - D. Thickest = thinner
  - E. Narrow = wide
  - F. Thin = thick
  - G. Highest = lowest
  - H. Heavier = lighter
  - I. More = less
  - J. Far = near
- II. Number concepts
  - A. Counting
    - 1. Rote
      - a. One's
      - b. Five's
      - c. Ten's
      - d. Two's
    - 2. Rational
      - a. One's
      - b. Five's
      - c. Ten's
      - d. Two's
    - 3. Identification— "Which group has four objects?"
    - 4. Reproduction— "Please give me five sticks"; "Draw a picture of three boats."
    - 5. Comparison— "How many more black boats than white boats are there?"
    - 6. Recognition of groups— "...able to identify number in a group without counting."
  - B. Numerals
    - 1. Recognition or identification of
    - 2. Writing of
  - C. Invariance or conservation of number
  - D. Ordinal property of number
  - E. Problem solving— verbal problems involving simple operations

    Example:

    2 + 2 = __________

    3 - 2 = __________
  - F. Recognition of terms
    - 1. Square
    - 2. Circle
    - 3. Triangle
    - 4. Rectangle

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G. Identification of coins
    1. Penny
    2. Nickel
    3. Dime
    4. Quarter
    5. Dollar
H. Relative value coins— "Which is more, three pennies or a nickel?"
I. Comprehension of fractional terms— 1/2, 1/4, 1/3.
J. Place value
K. Right versus left

MATERIALS AND METHODS FOR DEVELOPING MATHEMATICAL CONCEPTS

Having identified a list of appropriate concepts, the next question is how to go about teaching or developing these concepts. Unfortunately, words have no precise and absolute meaning. Teaching mathematics in the kindergarten or preschool may mean one thing to one person and something entirely different to another. The position taken here is that mathematics instruction at this level is not incidental, does not rely upon workbooks or papers and pencils, and is not to be confused with drill or the formal instruction frequently associated with traditional arithmetic instruction.

Instead, a good program is conceived of as consisting of a large number of widely varied experiences, each aimed at helping children develop specific concepts, ideas, or skills. While the activities may be extremely informal, the planning and preparation behind them should be done carefully and systematically. Games, concrete materials, manipulative devices, songs, stories, exploratory experiences, and planned activities make up the program. Teacher incentives include exploration, doubt, bafflement, and surprise in a climate where children are encouraged to take a calculated risk and receive support.

One of the first functions of the teacher is that of screening the children to determine which skills or concepts need developing or refining. His task then becomes one of selecting, creating, or devising experiences which lead the child to the desired understanding.

All of this is easier said than done, and perhaps the best way to convey an image of what is meant in the preceding paragraphs is to provide some illustrations. The following are offered, not as models to be imitated but as examples of activities that some teachers have found effective. An important point is that each activity is planned with specific children or groups of children and specific concepts in mind.

Let us say that a teacher has identified a group of children who can count by rote but cannot count rationally. He will then provide a large number of interesting experiences in which these children will count sets of blocks, tools, children, crayons, or other objects. In the beginning the teacher will count with them. The children may need to touch or pick up each object as it is counted in order to coordinate the thought processes with the physical processes.

One teacher working with four-year-old preschool children has five chairs at each table. When milk time comes, one child from each group has the privilege of distributing the proper numbers of napkins. One napkin is needed for each chair. There is much trial-and-error, much counting together, and much learning, for after a few months the same children can go to the kitchen and return with five cartons of milk, six cartons of chocolate milk, and four cartons of orange juice. The counting experiences in the room are not limited to milk and napkins. Other reasons for counting are used.

Another teacher, working with five-year-old kindergarten children, found some could count rationally to ten, to twenty, and some even further. Some were showing an awareness and recognition of the numerals (symbols), and others wanted to learn them. Having accepted the thesis that one can help children develop mathematical concepts without resorting to formal instruction, the teacher devised a set of experiences, including free play with some commercial materials made of wood or plastic and marketed under various names. The materials are designed so the children match objects or patterns with appropriate numerals. Some are interlocking and some may be put together only in consecutive order.

Further experiences with recognition of numerals were obtained through the use of a walk-on number line made of individual pieces of carpet on which the numerals had been spray-painted. These were laid end to end on the floor, and the children soon learned that their magic carpets didn't work if they were not in the proper order.

The teacher and children created "fun-type" of activities, such as the following: "Can you walk your age?" "Where am I?" "Walk forward two, walk back three; where are you?" "Can you walk the number of tables we have in the room?"

During the year the teacher made many other uses of the number line, but the ones described here developed skill in rational counting and recognition of the numerals. When not in use the individual pieces of carpet were stacked easily in a cupboard. Other walk-on number lines may be made from a variety of materials. They may be chalked, taped, or painted on the floor; or they may be purchased from commercial sources.

Several different stages or steps have been identified in the process through which a child learns to count. These are named in the list of concepts cited previously. The fourth stage is reproduction, when a child is able to "reproduce" a given quantity as in the directions, "draw a picture of five boats" or "bring me three sticks."

One teacher wished to give her group of preschool children continued experience with rational counting and, at the same time, focus on the reproduction aspect of counting. She cut triangles, rectangles, squares, and circles from felt. Scattering these around the room, the teacher started the game by saying, "Can you find two circles?" "Can you find one triangle and two squares?" In addition to providing experience with rational counting and reproduction, the game helped develop an awareness and knowledge of basic shapes. The teacher also used the game to foster recognition of colors, by saying, "Can you find one red triangle and one blue circle?' Other children were involved by being asked to verify the results.

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The invariance or conservation of quantity is a difficult concept for young children. A small child may be satisfied with a tall, narrow glass of milk, yet protest that he is being cheated when the same quantity is given to him in a wider, lower glass. He has not yet grasped the concept of conservation. Some teachers approach this concept by allowing children to experiment, using sand or water, with jars, bottles, and containers of various sizes and shapes.

All of the foregoing are simple, relatively informal, planned activities. Each of the examples involves the use of concrete materials and requires the "active" engagement of children. While many materials may be made by the teacher, an ever-increasing variety of helpful gadgets and equipment is available from commerical sources. Catalogues from companies such as Responsive Environments Corporation, Childcraft Equipment Company, Creative Playthings, and Judy Company list many useful materials.

SUMMARY

Our changing views on the nature of the development of a child's intellectual capacity have highlighted the crucialness of his early experiences. New programs are emerging which take this into consideration. In these, mathematical experiences have a role to play, both in the child's immediate world and as a foundation for later learning. The development of a sound and effective program seems to depend on certain essential or key steps:

1. Identification of concepts appropriate for and relevant to the children with whom one is working.
2. Screening and evaluation of children relative to each specific concept.
3. Selection or creation of games, materials, and experiences which help children develop specific concepts, that is, active engagement.
4. Following up with further evaluation and continued experiences as necessary.

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* From Math is a Verb, Schminke & Arnold, Dryden Press, 1971.

WHAT PIAGET SUGGESTS TO CLASSROOM TEACHERS*

Margaret Smart

A teacher stopped me in the hall to remark, "Now that I have experimented in my classroom with some of Piaget's intellectual tasks, never again will I look at children in quite the same way--and the change is for the better." Her comment reflected my own feelings because it's heady stuff to become a student of Piaget's work. Suddenly one finds it necessary to be an astute observer of children's behavior; a critical listener to their speech; an assessor of their thinking abilities; a stimulus for provoking their thought; and, indeed a player of a half dozen other roles, frequently all at once. Moreover, many long-valued teaching techniques appear trite and totally inappropriate in light of your new knowledge about children.

What is there in Piaget's studies which makes one perceive children in such different ways? After reflection, I deduced that three aspects of Piaget's work hold special meaning for me: (1) the ingenious experiments he has designed for revealing the nature of children's thinking; (2) his astute analyzations of children's intellectual development; (3) his provocative findings which imply classroom changes to promote more meaningful learning. Let us examine these three aspects separately.

Experiment: Conservation of Area

My introduction to Piaget's experiments occurred last year when I interviewed individually forty Mexican-American children between six and eight years of age. To replicate any of his experiments, however, it is necessary to understand his concept of conservation and its relationship to operational thought. Children conserve, Piaget tells us, when they recognize the equality or invariance of objects even if altered in shape, size or position. Prior to acquiring this ability, children's capacities for viewing things in their true relations are limited. They attend to only one attribute of an object or experience at a time and are unable to handle complex relationships or abstractions. The ability to conserve, then, is a prerequisite to the development of logical and systematic thought.

From among the many experiments Piaget designed to study children's acquisition of conservation, I chose one dealing with the composition of the area. His purpose (and mine) was to find out if young children "think of an area as a stable attribute which may be conserved even while the shape of an object is altered."[1]

The necessary materials consisted of two identical rectangular sheets of green cardboard, approximately 18" x 24"; two farmers (plastic); two cows (plastic); thirty wooden blocks 2" x 2" x ½"; and four wooden blocks 2" x 4" x ½".

Piaget's usual method of showing children an area (in this instance, one piece of cardboard), made up of sub-areas (the blocks grouped in one fashion on the cardboard), and then of modifying the visible structure of the total area (by taking the second sheet of cardboard and arranging in different positions an equal number of blocks on it) was replicated as the children looked on.

To stimulate the child's interest the interview followed a story pattern.

One of the farmers has decided to build a house on his field (a block is placed on field 1). Do the cows still have the same amount of grass?

Following the child's response, the interview continued:

The other farmer decides to build a house on his field as well (place a block in a different position on field 2). Will the cows have the same amount of grass to eat?

The next sequence introduced the central problem:

Each farmer decides to build another house (add a block to each field, spacing the houses on field 2 and placing alongside one another on field 1). Do the cows have the same amount of grass to eat?

From this point on each additional house placed on the fields left the child with the perceptual impression of more grass to eat for the first cow. It became increasingly difficult for the child to perceive the equality of the two fields as more houses were added.

The point at which he succumbed, if he did, to the perceptual differences required a change in procedural strategy. The spaced houses on field 2 were gradually moved closer together, while the child was asked, "What about now?" Some children would not admit equality until the blocks were arranged identically on both fields. Children were asked to explain their responses during the interview. What meaning did those replies have?

Analysis of Children's Responses

Piaget considered his results in the conservation of area task extremely informative. "Each new block which can be added without disturbing the recognition of equality," he wrote, "marks a step forward in development, until the point is reached when the solution is generated to all possible situations."[2] Some illustrations from my data will help perhaps to clarify Piaget's meaning.

> Six year old Tony was confronted with just two houses on each field.
>
> Tony: Field 1 has more grass because it has houses together and this one doesn't.
> Interviewer: What about now? (adding a third block to each field).
> Tony: No, because field 1 has more. It's not all covered (with houses).
> Interviewer: What about now? (moving spaced houses closer together).
> Tony: Yes, they're the same now because you moved them.

Piaget would label Tony at the intuitive level. Characteristically, the arrangement of the houses had to be identical for him to admit the equality of the remaining green spaces.

Then there were children who admitted the equality of the fields up to eight or ten houses, yet their language described their interest in the attributes and actions of the objects rather than in the task itself. For instance Gilbert, almost seven, agreed the fields were the same through twelve houses but he consistently reasoned their equality because "the grass is green." The attribute of color took precedence over any other aspects of the situation.

Six and a half year old Susie, admitting equality at eight houses, illustrated a growing mobility in her language: "Yes, the fields are the same because they're the same size." Similarly, other children evidenced their growing articulation with such remarks as:

"They don't look the same size."
"The fields are the same because there's two squares of grass."
"This is more spread out and this more squeezed up."

Gilbert, Susie and the others were all caught up in the perceptaul differences of the situation and vascillated from one aspect of the task to another. Still, the use of such words as size, square, look, spread out and squeezed up indicated a more mobile articulation.

Adelina, age six, was able to match five blocks on a one-to-one relationship. "Look," she remarked as she successively picked up a block from field 1 and placed it alongside its counterpart on field 2, "they fit on my fingers. They're the same." However, she reverted to perceptaul differences when she admitted, "This has more green (field 1) than this one." Her eyes continue to take precedence over her mind, although she is in transition from one level to another.

Seven and a half year old Mark has almost attained operational composition. Sixteen blocks were grouped on each field before he denied equality: "This field has more grass (field 1) but there's the same amount of blocks."

Eventually there does come a time when the children are no longer misled by any changes in position. Ramon, age seven and a half, remained confident throughout the interview. "The farmers," he responded, "are doing everything the same. They have grass the same and houses the same."

Six year old George also could not be shaken. "George," I queried, adding the seventeenth blocks, "are you watching what I'm doing with the blocks?" George, running his finger along a crack on the table, looked up and grinned: "No, but I know what you're doing. You're going to add one more block to each field."

Ramon and George relied upon an operational handling of the equality of the remainders. They were convinced of the necessity of their reasoning and thus were able to conserve area.

On the whole the children's responses fitted rather well Piaget's sequence for the development of operational thinking, but it is interesting to speculate why there were such discrepancies between their ages and their abilities to conserve. Why could six year old George conserve while Tony, his age counterpart, could operate only at the intuitive level? Piaget and others strongly suggest that the principle difference lies in the kinds of experiences children have. If this is true, what kinds of experiences should George, Tony and the others encounter if, as Piaget emphasized, "we wish to adapt teaching to the findings of developmental psychology as opposed to the logical bias of scholastic tradition."[3]

Time and Experience

Planning appropriate situations to fit the psychological findings suggest a different concept of time. How often have we heard, "Wait until the child is ready." "Give the child time to grow." It is true still that the rate at which a particular nervous system is developing affects the selection of experiences, but should time be the central determinant of developmental change? Rather, it seems to me, the experiences which occur in time are more important. Time nurtures the development of logical thinking because it permits laying in a stock of those situations necessary for children to accommodate new ideas and assimilate them into their own repertoires.

With such a concept of time, the choice of classroom experiences becomes crucial. In the classrooms from which my data were gathered, the experiences were designed to nurture the development of logical thinking. Special attention was given to fitting the appropriate experiences to the children's conceptual abilities, ranging from the intuitive level through the conservation stage.

In planning these experiences, Piaget's work suggested the importance of keeping in mind:

At the intuitive level young children are not able to deal with more than one property of an object or situation at a time.

Children able to conserve in one task may be unable to conserve with different materials or tasks.

Children tend to revert to the intuitive level when the demands of the environment are too removed from concrete content.

Experiences must be tied to action and accompanied by speech.

Examples from our classrooms illustrate each of the above four points.

First, a touch box served as one of the vehicles to promote children's use of analogy, which would consider the attributes of objects

Situation:

Children were taking turns feeling inside the touch box and guessing what was there (before school the teacher had placed a wooden soldier nutcracker inside).

Children:

It feels like a doll. No, it's like a monkey. I think it's a soldier. I think it's a man. No, it's like a coffeepot.

After all the children had guessed and the box was opened, Richard gleefully shouted, "I know what it is. It's a cracker's nut."

Teacher:*

(Her role is to model the language, provide the correct labels when needed, and reinforce positively the children's behavior). You are right, Richard. It is a nutcracker shaped like a wooden soldier. There were many good guesses, for it is like a doll, a soldier and a man.

The second point implied many related opportunities for the children to classify objects and discriminate between their differences and likenesses. During the many walking trips around the neighborhood the children looked for differences in fences, roofs, gardens and other aspects of their environment. One teacher walked her class to see the changes which occurred in the cottonwood trees during the fall, the winter and again in the spring. Floor maps were developed of the neighborhood. The children classified many objects as they talked about such attributes as softness vs. hardness or curved vs. flat surfaces.

Third, careful attention was paid to the educational setting. A variety of materials which would accommodate to many uses was used to extend the range of individual differences. For example: large floor blocks, table blocks, Lego blocks, play dough and, of course, all the art media were available for the children to manipulate and to work through their conceptualizing problems.

The final point has been described in the above illustrations because all the experiences permitted children to become actively involved and to learn what objects can do as they handle and discuss them.

It is hoped that these excerpts will suggest mapping out similar teaching strategies to promote children's intellectual growth.

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DIAGNOSING PUPIL ERRORS: LOOKING FOR PATTERNS*

Diagnostic teaching is teaching that looks at the errors children make and subsequently structures the learning experiences so that the errors will be eliminated. There is hardly a skill in the teacher's repertoire that is more important than the ability to identify pupil errors and prescribe appropriate remedial procedures.

The most fruitful diagnosis is the diagnosis that looks for patterns of errors rather than random errors. Teaching is so much more effective if a class of errors can be eliminated by a single remedial procedure. An examination of the errors children make on their papers reveals generally two kinds of errors: one type we may call careless errors and the other conceptual errors.

The careless error is generally spotted as the error that is not consistently made. It occurs most likely because the child loses track of where he is in a procedure. Algorithms that require the child to momentarily store information while he performs another operation typically result in this kind of error.

It is useful to distinguish the careless error from the conceptual error because the remedial procedures may be quite different. The careless error will probably be caught if the child reworks the problem. If the child is given a procedure for checking his work or for estimating the reasonableness of his solutions, then careless errors may be reduced. The conceptual error is quite different. No matter how many times the child works the problem, he will probably make the same mistake. We detect conceptual errors by seeing patterns of errors in a child's paper. The child may consistently make the error, which means that he has an incorrect understanding of the procedure; or he may make the error only sporadically, or only under certain circumstances, indicating that he has some grasp of the procedure but not complete understanding.

Diagnostic teaching requires that teachers (1) distinguish between conceptual and careless errors, (2) identify the precise nature of the careless errors, (3) infer the conceptual basis (cause) of the conceptual error, and finally (4) prescribe appropriate remedial procedures.

Figures 1, 2 and 3 are samples of the work of three children. A diagnostic analysis of each can be made.

DIAGNOSIS AND PRESCRIPTION

1. All Mike's errors are consistent ones, caused by a conceptual error; each one can be predicted. However, in D Mike also makes a careless error by reversing the digits in his answer.

2. When adding a two-digit number and a one-digit number, Mike first sums the ones column then adds to this sum the number in the tens column.

3. It is apparent that Mike does not understand the role of place-value in numerical representation. Neither is he disturbed by the impossibility of his answers.

4. The nature of his pattern of errors suggests that Mike needs much developmental work with concrete representations of numbers and with finding sums using concrete materials. Mike's work with symbols should be a record of what he finds with concrete materials. Base-ten blocks, bundles of sticks, place-value charts, or money (pennies and dimes) suggest themselves. Mike would also profit from instruction in estimating his answers.

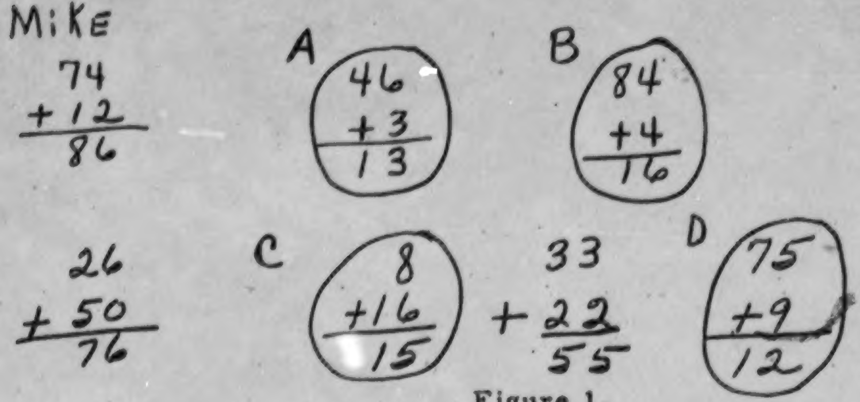
MiKE
74
+12
86
A
46
+3
13
B
84
+4
16
26
+50
76
C
8
+16
15
+ 33
22
55
D
75
+9
12

Figure 1.

DOUGLAS'S PAPER

1. The error that Douglas makes is consistent; it is a conceptual error.

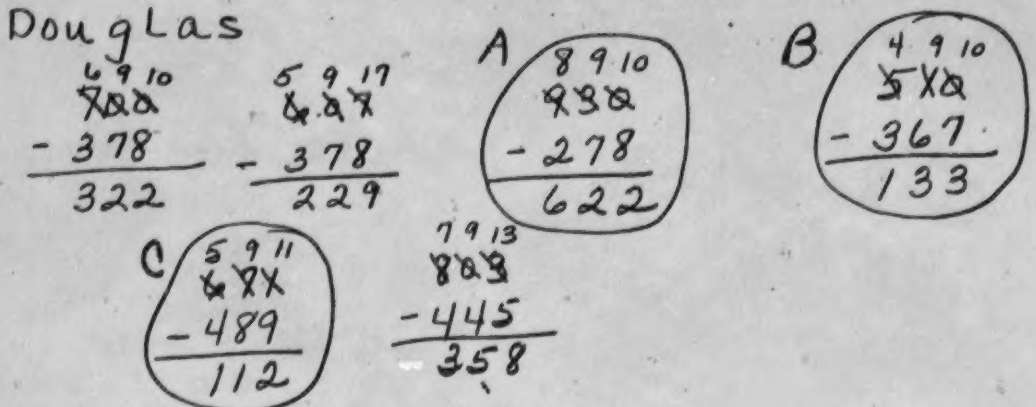
Douglas
6 9 10
700
- 378
322
5 9 17
607
- 378
229
A
8 9 10
900
- 278
622
B
4 9 10
510
- 367
133
C
5 9 11
601
- 489
112
7 9 13
803
- 445
358

Figure 2.

2. Notice that Douglas regroups hundreds to tens and always records the number as 9 tens, even when it is not appropriate. Notice also that sometimes it is appropriate.

3. Douglas does not understand the regrouping procedure in subtraction.

4. Having Douglas solve his problems with physical models of the place-value idea (such as base-ten blocks, bundles of sticks, place-value charts) will help him see the basis of the regrouping procedure in the subtraction algorithm. Giving him exercises in renaming will also help him, especially if accompanied by a physical model. Encouraging him to regroup 60 tens as 59 tens and 10 ones in-

4. (continued)....
stead of double regrouping would probably keep him from making so many errors of this kind.

SUSAN'S PAPER

1. Problems A and C are probably careless errors. Problems B and D are examples of a single conceptual error.

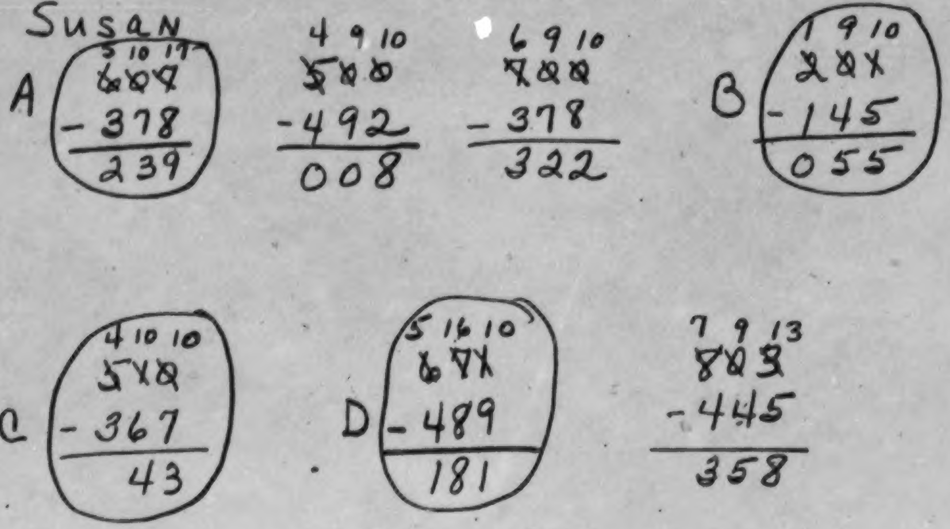

Figure 3.

2. In problem A, Susan did not completely record the regrouping of 10 tens to 9 tens and 10 ones. This is obviously not a consistent error. Look at all the occasions when she did not make this error. She probably recorded the 17 when she first encountered the regrouping problem but in the process of regrouping hundreds as tens forgot that she had not in fact done the regrouping yet (by modifying both tens and ones).

Problem C is probably also a careless error. She simply forgot to finish the problem.

3. Problems B and D are interesting. Susan did not add 10 ones to 1 one in problems having a 1 in the ones digit. She does not make this mistake when any other numeral occurs in the ones digit.

4. The basis of her problem seems to be some misunderstanding of the regrouping procedure in subtraction problems, at least in certain cases. Base-ten blocks, bundles of sticks, place-value charts, or money (dimes and pennies) could well exemplify the regrouping procedure in the subtraction algorithm.

Teachers who develop the practice of looking at children's papers critically to detect such patterns of errors are in a better position to provide for remediation of learning difficulties. They will probably also develop better teaching procedures for avoiding the very appearance of such errors in children's work. Children's papers provide vital information to the teacher. Happy is the child whose teacher is a good diagnostician!

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*This article excerpted from: Arithmetic Teacher, November, 1971. Pp. 467-469. Written by Tommie A. West.

TEST OF ARITHMETIC PRINCIPLES (Elem. Form)*

Part 1

1. Which of the following stands for the greatest number?

   a. 57,864
   b. 56,784
   c. 54,876
   d. 57,684

2. Which of the following also stands for 3467?

   a. 3 thousands + 4 hundreds + 6 tens + 7 ones
   b. 346 tens + 7 tens
   c. 34 hundreds + 67 tens
   d. 3 thousands + 467 hundreds

3. Which one of the following numbers has a 7 in the tens' place and a 6 in the ones' place?

   a. 4567
   b. 769
   c. 3776
   d. 9872

4. What would be the next two numbers in the following series?
   870, 880, 890, ____, ____

   a. 891, 892
   b. 892, 894
   c. 900, 901
   d. 900, 910

5. In the number 6,586,329,174 what is the fifth digit following the 5?

   a. 1
   b. 9
   c. 7
   d. 2

6. What is the third page in a book following page 109?

   a. 120
   b. 110
   c. 112
   d. 139

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*Welch and Edwards: Bulletin of School of Education, Indiana University, Vol. 41, No. 5, September, 1965.

7. When we count by ones, what are the next two numbers following 1009?

a. 1100, 1101
b. 1010, 1011
c. 1010, 1020
d. 1007, 1008

8. In 5284, what number is in the hundreds' place?

a. 5
b. 2
c. 4
d. 8

9. In the example at the right, the 4 really stands for how many? $\begin{array}{r} 4 \\ 8\overline{)356} \end{array}$

a. 4 tens
b. 4 hundreds
c. 4 ones
d. the problem must be completed to know the answer

10. When we count by two's what are the two numbers just before 92?

a. 94, 96
b. 92, 94
c. 90, 92
d. 88, 90

11. Which of the following numbers is equal to 15 thousands and 50 ones?

a. 1500050
b. 1550
c. 15050
d. 150050

12. How is the number six thousand thirteen written?

a. 60013
b. 6103
c. 6013
d. 600013

13. Which of the following is the same as three hundred forty-six?

a. 34 tens + 6 tens
b. 300 hundreds + 40 tens + 6 ones
c. 300 hundreds + 46 ones
d. none of these answers is the same

14. In the exercise at the right, it was necessary to "carry." How much was carried?

$$\begin{array}{r} 325 \\ \times\ 3 \\ \hline 975 \end{array}$$

a. a hundred
b. a one
c. a seven
d. a ten

15. In the number 4231, the 2 stands for a value how many times as great as the value the 1 stands for?

a. 2
b. 200
c. 1/2
d. 1/4

16. Which of the following numbers is 100 more than 99,999?

a. 100,099
b. 100,000
c. 100,009
d. 99,000

17. The number one hundred four is equal to how many ones?

a. 1
b. 00
c. 4
d. 104

18. When counting by three's, what is the seventh number following 21?

a. 39
b. 45
c. 42
d. 36

19. How is the number 237,058 read?

a. two hundred thrity-seven thousand, five hundred eight
b. two hundred thousand, thirty-seven, and fifty-eight
c. two hundred thirty-seven and fifty-eight
d. two hundred thirty-seven thousand, fifty eight

20. How is the number four million, four hundred seventy-two thousand, twenty written with numerals?

a. 4,472,000,020
b. 4,400,072,020
c. 4,472,020
d. 4,472,200

21. Which of the following is in the number 26,785?

a. 2 ten thousands
b. 8 ten thousands
c. 785 hundreds
d. 26 ten thousands

22. Which of the following is the largest number?

a. .064
b. .6
c. .046
d. .46

23. Which of the following would be true if the answers to these two exercises were found?

A. $$\begin{array}{r} 38 \\ 29 \\ +\ 7 \\ \hline \end{array}$$

B. $$\begin{array}{r} 380 \\ 290 \\ +\ 70 \\ \hline \end{array}$$

a. the answers would be the same
b. the answer to "B" would be 10 times as great as the answer to "A"
c. the answer to "B" would be 10 more than the answer to "A"
d. the answer to "A" would be greater than the answer to "B"

24. How much of the three tens is "borrowed" or changed in this subtraction?

$$\begin{array}{r} 36 \\ -19 \\ \hline \end{array}$$

a. 1 ten
b. 1 one
c. 16 ones
d. the problem must be completed to know the answer

25. How should .359 be read?

a. three hundred fifty-nine
b. three hundred fifty-nine tenths
c. three hundred fifty-nine hundredths
d. three hundred fifty-nine thousandths

26. In the example at the right, what is the reason for placing the 66 one place to the left?

a. because the 66 is really 660
b. because the second 6 must be under the 2
c. because this is the rule in multiplication
d. none of these reasons is correct

$$\begin{array}{r} 33 \\ \times\ 21 \\ \hline 33 \\ 66 \\ \hline 693 \end{array}$$

27. Which of the following is a way of writing nineteen hundredths?

a. 1900
b. .0019
c. .19
d. 19.00

28. Which of the following is the smallest number?

a. .101
b. .11
c. .011
d. .1

Part II

1. How would the sum in the following example be changed if the 53 were placed above the 88?

a. the sum would be the same
b. the sum would be greater
c. the sum would be less
d. it is not possible to tell until both sums are found and compared

2. These statements are true:

4 1/2 x 3 = 3 x 4 1/2
9 1/4 x 8 = 8 x 9 1/4

which of the following is not true?

a. 5 x 6 1/2 = 6 1/2 x 5
b. 7 x 8 1/4 = 8 1/4 x 7
c. 341 x 205 = 205 x 341
d. 5 1/4 x 19 = 19 1/4 x 5

3. Which of the following will give the same answer as 15 x 38?

a. 15 x 40 - 15
b. 10 x 38 + 5 x 38
c. 38 x 10 + 5
d. 10 x 30 + 10 x 8 + 5 x 3 + 5 x 8

4. Which of the following is the same kind of problem as 3 x 4 = 4 x 3?

   a. 5 x 3 = 15
   b. 3 x 5 = 15
   c. 3 x (2 + 3) = 15
   d. a x b = b x a

5. In the expression a x b x c, how will the answer be changed if it is written in this way: c x b x a?

   a. the answer will be less
   b. the answer will be greater
   c. the answer will be the same
   d. it is not possible to tell until the answer is found both ways

6. If you know these statements are true:

   4 + 5 = ⧗   5 + 5 = θ   4 + 4 = ⊗   5 + 4 = Ɏ

   which of the following is also true?

   a. ⧗ = θ

   b. ⊗ = Ɏ

   c. ⊗ = ⧗

   d. ⧗ = Ɏ

7. These show a certain rule in arithmetic:

   3 x 4 = 4 x 3   5 x 4 = 4 x 5

   which of the following shows the same rule?

   a. 6 x 7 = 42
   b. 6 x 7 = (3 + 3) x 7
   c. 6 x 7 = 7 x 6
   d. none of these answers shows the same rule

8. ♁ + θ = ∀̸
   ⧗̄ + θ = Ɏ
   ⧗̄ + ⧗̸ = ⊗

   If you know the above statements are true, which of the following is also true?

   a. θ + ∀̸ = ♁
   b. θ + ♁ = ∀̸
   c. θ + θ = Ɏ
   d. none of these answers is true

9. If b + a = b, what value does a have?

a. one
b. zero
c. "b"
d. it is not possible to tell

10. You know that 8 x 9 x 12 = (8 x 9) x 12 = 72 x 12 is true; which of the following shows the same rule?

a. 8 x 9 x 12 = 8 x (9 x 12) = 8 x 108
b. 12 = 6 x 2 = 3 x 2 x 2
c. 12 = (10 + 2) = (5 + 5) + 2
d. 12 = (12 x 1) = 3 x 4 x 1

11. How do you find the product of x, y, and z?

a. multiply x by y and then multiply by z
b. multiply x by z and then multiply by y
c. multiply y by z and then multiply by x
d. each of these answers is correct

12. In the exercise b - c = b, what value does the c have?

a. one
b. zero
c. "b"
d. it is not possible to tell

13. In which of the following has the fraction $\frac{9}{12}$ been divided by one?

a. $\frac{9}{12}$ b. $\frac{8}{12}$ c. $\frac{3}{4}$ d. $\frac{9}{11}$

14. You know that 30 + 66 + 46 can be added in this way: (30 + 66) + 46 = 96 + 46. Which of the following shows the same rule for addition?

a. 30 + 66 + 46 = (30 + 60) + (40 + 12) = 90 + 52
b. 30 + 66 + 46 = 30 + (66 + 46) = 30 + 112
c. 30 + 66 + 46 = (30 + 70) + 40 = 100 + 40
d. none of these shows the same rule

15. If you know that a + b = c and that d does not equal one or zero, which of the following is true?

a. a + b + d = c
b. a + b = c + d
c. (a + b) x d = c
d. c x d = a x d + b x d

16. These statements show a certain rule:

$3(5 + 6) = 3 \times 5 + 3 \times 6 = 15 + 18 = 33$

$8(2 + 3) = 8 \times 2 + 8 \times 3 = 16 + 24 = 40$

Which of the following shows the same rule?

a. $8 \times 7 = 7 \times 8$
b. $5 \times 15 = 5(10 + 5) = 50 + 25 = 75$
c. $5 \times 4 + 3 \times 2 = 8 \times 6 = 48$
d. none of these shows the same rule

17. If you know ɏ = ѿ + ⧗ , and that θ does not equal zero, which of the following is true?

a. ɏ − ⧗ = θ + ѿ
b. ɏ + θ = ѿ + ⧗
c. ɏ × θ = (ѿ + ⧗) × θ = ѿ × θ + ⧗ × θ
d. none of these is a true statement

18. These statements show a rule for addition:

$2 + 7 + 5 = (2 + 7) + 5 = 9 + 5 = 14$, and
$2 + 7 + 5 = 2 + (7 + 5) = 2 + 12 = 14$ also

Which of the following shows the same rule?

a. $4 + 3 + 5 = (4 + 3) + 5 = 7 + 5 = 12$, and
$4 + (3 + 5) = 4 + 8 = 12$ also

b. $4 + 3 + 5 = 12$ and $4 + 3 + 5 = (4 + 2) = (1 + 5) = 6 + 6 = 12$ also

c. $4 + 3 + 5 = 12$ and $5 + 3 + 4 = 12$ also

d. none of these shows the same rule

19. Which of the following is not a true statement?

a. $15 \times 18 = 18 \times 10 + 18 \times 5$
b. $15 \times 18 - 15 \times 10 + 15 \times 8$
c. $15 \times 18 = 15 \times 20 - 15 \times 2$
d. $15 \times 18 = 3 \times 18 + 5 \times 18$

20. In the problem: $\begin{array}{r} ab \\ + \ rm \\ \hline cb \end{array}$ which of the following is true?

a. $a + b = m$
b. $a + m = b$
c. $a + m = r$
d. $a + m = a$

Notes about the Welch and Edwards test:

The test was standardized on 1,500 children in grades 3, 4, 5 and 6 in Indiana. The reportedly reliability for the test is 0.88, with lesser reliabilities for each of the four grades tested. Its face validity was attested by a panel of twelve elementary school mathematics educators. Presented in Figure 1 is the dispersion of the 48 items which comprise the test according to the concepts they treat.

Concept	Number of Items
place value	17
ordinality	6
interpreting numerals	5
commutative property	6
associative property	5
distributive property	5
identity element	4
	48

Figure 1

The test is organized into two parts. The first section includes items dealing with the Hindu-Arabic notation system. The second section is devoted to items related to the basic postulates, including the commutative property with respect to addition and multiplication, the associative property with respect to addition and multiplication; the distributive property of multiplication over addition, and the respective identity elements for addition and multiplication.

Some aspects of new elementary mathematics are not measured effectively by this and other tests:

synthesis of knowledge
discovery
deductive logic
system thinking
applied mathematics
algebraic flexibility
geometry
principles of measurement

PIAGET'S THEORY OF DEVELOPMENT WITH IMPLICATIONS FOR TEACHING ELEMENTARY SCHOOL MATHEMATICS*

Anthony J. Picard

The work of Jean Piaget is causing many changes in the theory of learning. One area in which these changes are being felt and in which the Swiss psychologist has personally conducted experiments is the area of the learning of mathematical concepts by children. This paper will review the theory of the development of knowledge as proposed by Piaget and then consider the implications of this theory for the teaching of mathematics in the elementary school.

The concept which Piaget considers fundamental to the development of knowledge is that of an operation. An operation is defined as "a set of actions modifying the object and enabling the knower to get at the structure of the transformation." (1)[1] This definition requires the learner to carry out his actions on an object or a set of objects and to be cognizant of the nature of the change he has produced. That is, he must be able to "reason" about the actions he is performing. Examples of operations would be:

1. ordering objects (according to size, weight, age, color or other specific criteria).

2. putting objects in a series (by following a pattern or constructing a hierarchy).

3. counting objects (by setting up a one-to-one correspondence or by recognizing cardinality).

4. measuring objects (of two and three dimensions using a homemade unit or a standard one).

5. classifying objects (by partitioning a set into subsets based on some property; by arranging sets into a hierarchy).

The criteria for determining whether a particular behavior is an operation in terms of the definition above are twofold. First the action must be interiorized to the learner. That is, it is initiated and sustained by the learner and not some external agent. For example, the young child who is taught to repeat "wun", "too", "three", etc., may not be counting. "Too" may only be the sound between "wun"

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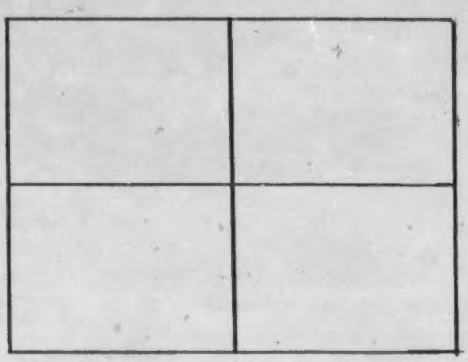
[1] Number in brackets refers to bibliography.

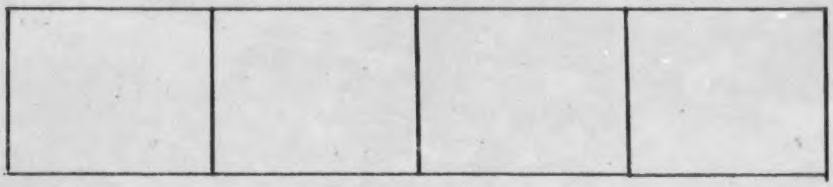
*From: School Science and Mathematics, April, 1969; pp. 275-80.

and "three." If the child (1) does not recognize the common property of the set consisting of his parents, the set consisting of his hands, and the set consisting of his shoes (cardinal number), (2) cannot match the objects in one collection with those of another (one to one correspondence), (3) cannot distinguish a set with more objects from one with fewer objects (ordering), (4) does not realize that the number of objects in a collection remains fixed in spite of their arrangement (conservation of number), then he has not interiorized counting.

The second criteria is that the action must be reversible. That is, the learner must be aware of a second action which when applied to the object following his first action restores the object to its original condition. A simple test for reversibility consists of four squares arranged into a larger square.



The child is asked to arrange the squares into a straight line without overlapping.



He is then asked if the new figure takes up more "space", less "space", or the same "space" as the first figure and why he thinks so. If he says the new figure takes up more "space" because it is longer or less "space" because it is thinner, he has not achieved reversibility. If he says it takes up the same "space" because the second figure can be rearranged to form the first one, he has achieved reversibility.

It is important to keep in mind the fact that an operation is never isolated; it is always linked to another operation with the framework of a total structure. An investigation of the examples of operations given earlier will illustrate this point. Ordering objects according to some criteria may involve putting objects in a series or constructing a hierarchy. Counting objects may involve ordering objects or putting them in a series. Putting objects in a series may involve ordering and classifying objects.

Piaget lists four stages through which the learner proceeds in the development of knowledge. These stages are based upon the development, in the learner, of a well defined set of operations.

The first stage is the sensory-motor which usually lasts from birth to the age of two years. All knowledge is built on the fundamental learning of this period. For example, the child learns physical objects have a permanence independent of his perception. This is evidenced by the fact that very young children do not hunt for an object once it is out of sight. As they grow older, they learn to look for an object which has been hidden. The child also learns physical motions have a cause and his own actions may produce a chain of events.

The second or pre-operational stage which usually lasts from age two to age seven is characterized by the development of language and the use of symbolic referents for objects. But the criteria for operations are still not satisfied. "Thought at this stage is largely based on perception, and usually one aspect, dimension, or relation is considered at the expense of others." (2, p. 40). For example, at this stage if a pile of coins is placed on a table and then the coins are spread out, a child will say there are more coins on the table after they have been spread. The criterion for reversibility has not been satisfied.

The third stage is the concrete operational and usually occurs from age seven to age eleven. The child is able to perform true operations at this stage. Classifying or ordering are examples but these operations are carried out in terms of concrete physical objects. At this time the idea of number, the idea of spatial and temporal relations, the fundamental operations of elementary arithmetic, elementary geometry, and even elementary physics make their appearance. "During the concrete operational stage, the child develops number concepts in terms of a collection.....substance, weight, and volume. They are always discovered in this order: substance, weight, volume" (2, p. 41).

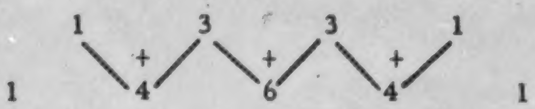
The final stage or formal operational usually occurs from age twelve to age fifteen. The adolescent is no longer limited to the manipulation of physical objects in this stage. He can successfully carry out his operations on abstract ideas. He is able to reason from hypotheses rather than from objects. He is even able to construct new knowledge by reorganizing knowledge he already possesses. Mathematically, he is capable of understanding and constructing chains of deductive reasoning.

Piaget stresses the order of appearance of these four stages is fixed but the "timetable" suggested for their appearance varies with the individual learner. Each succeeding level of development involves a structuring of elements the child did not see as related in the previous stages.

Four factors contribute to the development of knowledge as proposed by Piaget: nervous maturation, encounters with experience, social transmission and equilibration or auto-regulation (3). The child is mentally passive with respect to the first three factors. He is, in effect, receiving information from them but he is merely storing it. Suppose, for example, the child is presented the following array of numbers and asked to find as many patterns as he can.

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

He may notice each row begins and ends with 1. He may notice, moving from left to right or from right to left, the numbers in any row increase and then decrease. He may notice a subset of the counting numbers in natural order 1, 2, 3, 4, 5. He may notice any row can be formed from the row above it in the following manner:



The number of patterns the child discovers and their complexity depends on the amount of experience he has in looking for number patterns. If another child shows him a pattern, he is receiving information by social transmission.

With respect to the fourth factor, the child is mentally active. He is attempting to assimilate and accommodate the information from the first three factors into his existing structure of knowledge. This may involve a change in existing structures if the information is not a radical departure from the view he currently holds. It may involve abandoning a part of the structure if the departure is extreme. In either case, development results from the individual's attempts at coordinating partial understandings. A child who has learned an algorithm for problems such as $\begin{array}{r}27\\-14\\\hline\end{array}$ will have to change the structure of that algorithm or develop a new one to solve problems such as $\begin{array}{r}27.\\-19\\\hline\end{array}$

A simple analogy at this point will help to explain the role these four factors play in the development of knowledge. A pile of stones and some mortar do not of themselves form a wall. The stones must be arranged properly and the mortar used to keep them in place before the wall can take shape. The first three factors (nervous maturation, encounters with experience and social transmission) may be thought of as supplying the child with the raw materials for the development of knowledge; the stones and mortar. The fourth factor (equilibration) corresponds to the process of arranging the stones and applying the mortar; the construction by the child of a "wall" of knowledge.

What does all of this have to do with teaching mathematics in the elementary school?

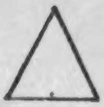
First, most children in the elementary school are at the pre-operational or concrete operational level of development. They must therefore be given ample opportunity to carry out their mathematical operations with actual physical objects. They should count, order, arrange, add, or subtract real oranges, or apples, or toys. The textbook may offer attractive pictures to look at, but these should not be substituted for reality until the operations on physical objects have been mastered by the child.

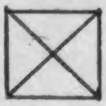
Second, learning doesn't result from talking to the child. The child must be actively involved in creating the mathematics he is to learn. Symbols for exponents, variables, operations, and functions should be invented by the child. Operational definitions for geometric figures should be constructed by the child and refined as he sees the need for refinement. Tentative generalizations based on observation should be formulated and altered as counterexamples appear.

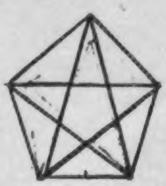
Third, since one of the factors affecting the development of knowledge is social transmission, it is important for the child to compare his answers with those of other children. As a result, he comes to accept the possibility of more than one solution to a problem. Unique or creative solutions are something he can look for each time he approaches a problem. He also comes to accept the possibility a situation may be viewed in different ways. Seemingly unrelated problems can now be grouped together and a general solution developed for that class of problems. Suppose a child is finding the perimeter of a square by using the formula $p = 4s$ (where s is the length of a side). Suppose another child is finding the circumference of a circle by using the formula $C = \pi r$ (where r is the length of the radius). They will both have a deeper understanding of direct variation after comparing their data and realizing both sets of data may be described by the simple mathematical statement, $Y = kx$ (where k is a constant).

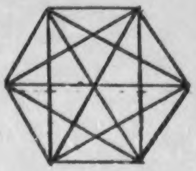
Fourth, there should be many opportunities when the child is presented with collections of mathematical data such as:

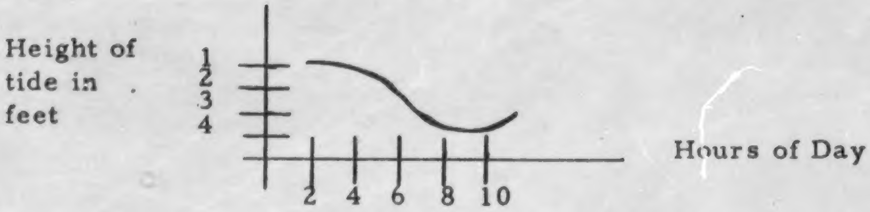
Sets of numbers e.g., 9, 18, 27, 36, 45, 54, 63, 72, 81, 90
Geometric shapes (as below)











He should be asked to list everything he observes about the data. In effect, he must be allowed to draw his own generalizations, simple or profound, without being told to look for specific ones.

Fifth, after determining the child's mathematical strengths and weaknesses, it is necessary to determine his stage of development as defined by Piaget. Then educational experiences appropriate for that stage of development could be prescribed for him. It is ridiculous to try to teach a child "3 + 2 = 5" if he can't determine when he has five objects on his desk.

Sixth, since the concrete precedes the formal operation, new content should be based on experience with physical objects. This applies not only to the young child, but to the older child as well. Symbolic statements, of number properties such as the commutative law, of space properties such as the relationship among the number of surfaces, vertices, and edges of a cube, or of algebraic properties such as the structure of a group, take on fuller meaning if the child has had several opportunities to work out the generalization with concrete objects.

Finally, since the factors of maturation, experience and social transmission can be partially controlled, it may be possible to speed up the stages of development. Piaget himself believes the process of development unfolds at a steady pace and little can be done to accelerate it (4, p. 30-31). Those educators who reject this position point out "many of Piaget's experiments were conducted during the 1930's in Switzerland. Toys available today in many American homes may influence the outcomes of some of his experiments" (5, p. 49).

If acceleration of development is possible, we can profitably expose the child to more mathematics than is presently incorporated into the elementary school program. The crucial problem then becomes one of deciding what mathematics the child should be exposed to and what order of presentation will produce the greatest mathematical maturity.

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TESTING READINESS*

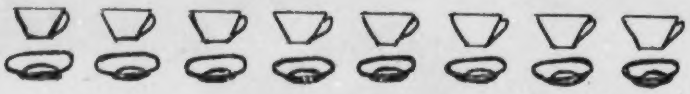
The research of Jean Piaget has shown that many children age seven and eight are still in a transitional stage between intuitive thinking, where things really are what they seem to be and the stage of concrete operations, where children naturally resort to physical experiments to determine truths about situations. For example, a child at an "intuitive" stage (shown below in the first experiment) may be content to say the longer line has more objects in it, whereas the child who has reached the stage of concrete operations will move the objects about to check his theory that each line contains the same number of objects. It is important for teachers to know what stage their children have reached so that they may provide suitable experiences for them. Experiments with measuring length are rather meaningless if they are conducted by a child for whom length is subjective, and for whom a stick is longer when it looks longer and shorter when it appears shorter.

During the past two decades the tests described here have been performed thousands of times with many children all over the world. They are very easy to do with individual children in a normal classroom setting and require only the simplest materials. Every teacher of second or third grade children should use these tests to check up on the developmental stage of children in her class whom she suspects of mathematical immaturity.

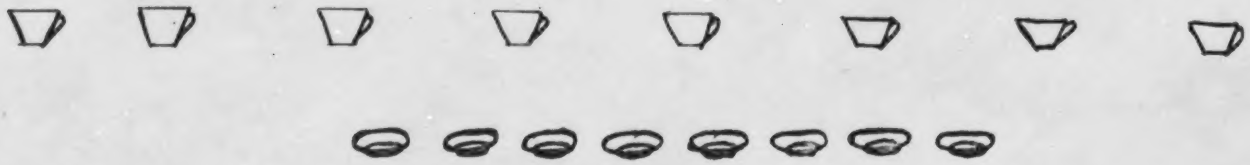
EXPERIMENT 1: CONSERVATION OF NUMBER

Eight red buttons and eight black buttons (or any two other sets of eight objects, such as eight cups and eight saucers) are needed for this experiment. Simply line up the two sets so that the child being tested can see that they match one-to-one. Ask the child if there is the same number of, say, cups and saucers. You should expect an affirmative answer to this. Now take one set, say the cups, and spread the parts out so that they take up more space than the other set and repeat the same question. "Are there the same number of cups as saucers? More? Less?"

Arrangement 1:



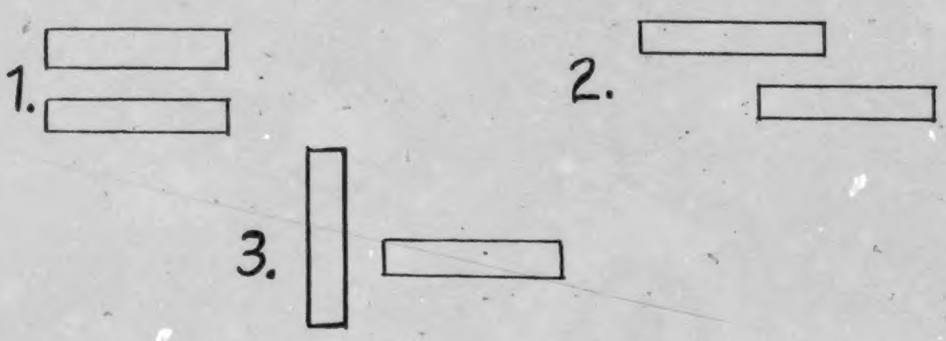
Arrangement 2:



If a child insists that there are more cups than saucers when they are spread out, he may still be in the intuitive stage. Of course, you must be careful not to tip off the child being tested by inflections in your question or facial expression. Play poker. Do not try "to teach" at this point; you are trying to determine what he knows now, not what he can "seem" to learn.

EXPERIMENT 2: CONSERVATION OF LENGTH

In this experiment you use two sticks of the same length. Place them side-by-side. Ask, "Are they the same length?" Now slide one to the right and ask the same question. Now place one in a vertical, the other in a horizontal position, and ask again if they are the same length.



The child who is not sure that the sticks maintain the same length as they are moved about may not have developed the idea of conservation of length.

EXPERIMENT 3: CONSERVATION OF VOLUME

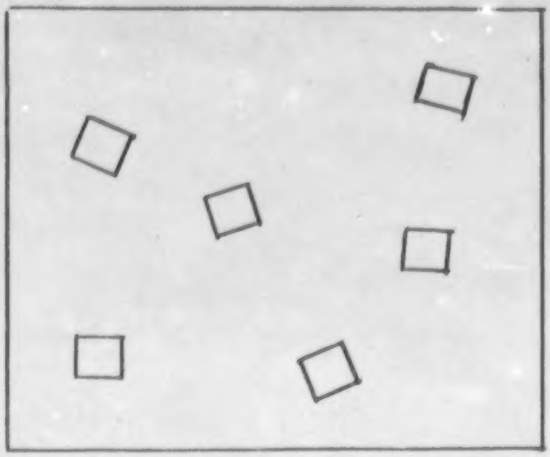
Add some coloring to water so that it will be quite visible through clear glass. Use two glass jars, one tall and thin, the other short and fat. Pour some water into the short, fat jar. Then pour that water into the tall, thin jar. Ask the child if there is the same amount of water in the tall, thin jar as there was in the short, fat one. After you have poured the water back and forth several times and asked the question, you will have an idea of that child's concept of the conservation of volume.

EXPERIMENT 4: CONSERVATION OF SUBSTANCE

Get a piece of Plasticine or clay and divide it into two equal pieces. If the child being tested thinks one of the pieces is bigger than the other, have him readjust them until he is convinced that they are of equal size. Now roll one of the lumps of clay into a long sausage shape, and leave the other as a ball. Ask, "Are they the same? Is one bigger? Which one?" The child who is unsure or thinks that the one that looks bigger is bigger may still be thinking intuitively.

EXPERIMENT 5: CONSERVATION OF AREA

A piece of green art paper is a good background for this experiment. You will also need some shapes of a contrasting color; cubes will do. Place six cubes packed together on the green paper. Tell the child being tested that the cubes are houses and that the green paper is grass. Now spread the six cubes apart on the paper. Ask the child to compare. "Is there more grass now? Less? The same as before?" If the child is unsure, he may not have the concept of conservation of area.



None of these experiments should be taken as definitive, only as indicative that the child may still be in the intuitive stage--or at least that he is able to slide back into it easily.

Any second or third grade teacher who has not yet tried any of these Piagetian experiments has quite a surprise in store. Imagine how much understanding a child will have of the normal classwork with number combinations if he believes that the number of objects in a set changes if the set is rearranged! Elementary education owes a profound debt to the work of Jean Piaget.

DIAGNOSIS:
REFLECTIONS ON THE C/SC/A* CONTINUUM#

The usual definition of concrete as real objects and semi-concrete as pictures and diagrams seem to be inadequate as a means of communicating certain major methodological concepts to future teachers. Such a definition is too broad in the types of learning experiences or reinforcements which they provide.

Concrete learning experiences refer to child manipulative experiences. Inherent in the use of the words is the assumption that within the learning experience the child is having tactile and/or kinesthetic reinforcement of the concept under study. It should be noted, however, that the use of such a definition is also distinctly different from mere activity or involvement on the part of the child. The child learns a concept or practices a concept by acting on objects in such a way as to perform operations on objects which exemplify concrete manifestations of concepts.

A. If a child makes sets of two blocks and three blocks and then points to each one in turn to decide that the total is five, he is having a concrete learning experience, related to addition.

   1. He is acting (movement of blocks.).
   2. He is using concrete materials (blocks).
   3. He is "acting out" a concept (addition).

B. If a child has a set of marbles and a set of jacks and pairs them in the partitions of an egg carton, he is having a concrete learning experience related to one-to-one correspondence.

   1. He is acting (pairing marbles and jacks).
   2. He is using concrete materials (marbles and jacks).
   3. He is "acting out" a concept (one-to-one correspondence).

C. If a child has 100-square regions of graph paper, 10-square strips of graph paper and unit-square regions of graph paper, he can use them to add decimal fractions as a concrete experience.

   1. He is acting (moving graph paper regions).
   2. He is using concrete materials (graph paper regions).
   3. He is "acting out" a concept (addition of decimal fractions).

---

* C/SC/A refers to Concrete/Semi-Concrete/Abstract.

#From: Robert Underhill; Teaching Elementary School Mathematics, Professional Supplement, Charles E. Merrill Co., 1972. Pp. 99-103.

D. If the teacher demonstrates with objects, the child is not having a concrete learning experience under this definition.

1. The child is not acting.
2. The child is not using concrete materials.
3. The child is not "acting out" a concept.

E. If the child uses a paper-and-pencil number line to determine answers to multiplication sentences such as 4 x 3 = ?, he is not having a concrete learning experience.

1. The child is acting (movement on the number line).
2. The child is not using concrete materials. (There are not twelve objects.)
3. The child is "acting out" a concept (multiplications as jumps on a number line).

F. If the child uses a walk-on number line to decide if 3 5 or 5 3, he is having a concrete learning experience.

1. The child is acting (movement on the number line).
2. The child is using concrete materials (himself, kinesthetic reinforcement to feel that three comes before five).
3. The child is "acting out" a concept (inequalities translated into more or fewer steps).

Of the six illustrations cited, A, B, C and F are concrete learning experiences. D and E are not.

Semi-concrete reinforcement of concepts can be intuitively defined as learning experiences which are predominantly visual in nature. The child observes pictures, diagrams or demonstrations. Such visual reinforcement can be static or it can involve movement, i.e., the presentation could be a picture or diagram or slide or filmstrip representation, or the teacher may demonstrate a concept, a movie can be viewed, or the child may draw a picture. However, the child does not move objects to learn the concepts.

Abstract experiences, on the other hand, involve no pictures, diagrams or objects. They are usually best described as symbol manipulations. Algorithms without concrete or semi-concrete referents are abstract. Mathematics sentences related to set operations on letters which are themselves elements of sets or which represent points, lines and rays are abstract if no diagrams are used.

In summary, a concrete learning experience is primarily tactile and/or kinesthetic, a semi-concrete learning experience is primarily visual, and an abstract experience can best be described as symbol manipulation.

REFLECTIONS ON DIAGNOSTIC INSTRUCTION

Brueckner's definitions can be roughly paraphrased as follows: A survey test is one which examines several major concepts to ascertain the general level of functioning of an individual or group of individuals. Broad areas of weakness can be determined. An analytical test is one which examines in detail the performance of an individual or group of individuals on one major concept such as addition of whole numbers or multiplication of non-negative rational numbers; a clinical test is designed to probe the deficiencies of a single child in relation to one major concept by using interview and observation techniques in addition to formal testing.

Brueckner spent many years developing the details of his approach to diagnosis and derived several extensive lists of errors from analyzing children's work.

His theory is outdated. He was a leader in diagnostic instruction in mathematics, but his theory fails to take into account two major ideas of mathematics education of the second half of the twentieth century: (1) "Understanding" or "meaning" is a major goal of the mathematics curriculum, and (2) Children learn through a succession of concrete, semi-concrete, abstract encounters with mathematics concepts.

The emphasis in Brueckner's theory was on computation. His influence is seen today in nearly every standardized test on basic skills. Nearly all the evaluation of mathematics achievement is done at the abstract level and much of it can be done by rote.

The text which this manual accompanies contains the nucleus of an approach to diagnostic instruction which is more in line with current thinking in mathematics education. Brueckner's terminology has been retained, but the definitions have been modified.

Survey Test--same as Brueckner's definition.
Analytical--same scope as Brueckner's
    Group--evaluates semi-concretely and abstractly
    Individual--Evaluates concretely, semi-concretely and abstractly.
Clinical--an individual analytical test plus interviewing/observation information.

The survey test is still defined to be a general test which evaluates pupil performance on several major concepts.

The analytical test is constructed by breaking down one major concept into it's subconcepts. Group tests do not include concrete questions because the teacher cannot observe several children's manipulations simultaneously. The group test contains six questions for each subconcept as suggested by Brueckner (three semi-concrete questions and three abstract questions). The individual test also contains six questions for each subconcept (two concrete, two semi-concrete, two abstract). Three questions at each level are not deemed necessary because the test is an individual test and the mastery should be evident in the completion of two exercises at each level.

The clinical test is used with those children who progress is uncommonly slow. The clinical test is used in those special cases in which test performance alone is inadequate in establishing a program of remediation for a given child. Details of the child's home life, like style, attitudes, self-concept and so on may be needed to complete an adequate diagnosis.

This nucleus of a theory of diagnosis for elementary school mathematics is much more painstaking in its attention to current goals of mathematics instruction, theories of child development and theories of concept acquisition. As in nearly every other phase of human existence, diagnostic theory must continue to evolve into newer forms which reflect current trends and new knowledge.

A MODEL FOR USING DIAGNOSIS IN INDIVIDUALIZING MATHEMATICS INSTRUCTION IN THE ELEMENTARY SCHOOL CLASSROOM*

A revolution in mathematics instruction has begun and will probably continue through this decade. The revolution consists of a major change in the elementary school teacher's role in the educative process. The teacher of elementary school mathematics is modifying his traditional role as "director" and "lesson planner" for classroom-sized groups and is assuming the role of "instructional programmer" for individual learners. In assuming this role, the elementary mathematics teacher's concern has shifted to exercising more control over the instructional environment by arranging scope, sequence, content, feedback, evaluation, and materials appropriate for individual learners.

The acceptance of this new role in the educative process has brought about a series of other changes. These include a change in the elementary mathematics teacher's conception of instruction and the development of various types of individualized instruction. The purpose of this paper is to (1) present an overview of the two changes just mentioned, and (2) justify the need to modify and present a model for modifying the way in which elementary teachers conceptualize and use diagnosis to individualize mathematics instruction in their classrooms. The film, Using Diagnosis in the Mathematics Classroom[1], illustrates how the suggested model of diagnosis could be used to individualize mathematics instruction in an elementary school classroom.

ELEMENTARY TEACHERS' CHANGING CONCEPTION OF INSTRUCTION

Because of the teachers' role shift, there has been a corresponding change in the way in which elementary mathematics teachers are being encouraged to think about instruction. Traditionally, mathematics educators, school administrators, and mathematics supervisors have encouraged teachers to seek out and try new materials, techniques, and procedures. In response to this encouragement, elementary teachers have attempted to improve their teaching performance by using new materials and techniques with individuals, small groups, and the class. Hence, the elementary teachers' primary interest naturally became focused on the ways and means of changing their teaching performance.

As a result of this interest, the key question asked by most elementary teachers became "What new instructional procedures should I adopt to improve my teaching performance?" rather than "What should or can my students learn?" These two questions are obviously related, but under the means, or teaching-process, orientation just described the direct connection was obscured. Thus it is not surprising that most elementary teachers' conception of instruction became and has remained, until recently, means, or teaching-process, oriented.

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*Excerpted from: Arithmetic Teacher, November, 1971--Cecil R. Trueblood. Pp. 505-511.

[1] This is one of a set of five films developed at The Pennsylvania State University for the Interpretive Study of Research and Development in Elementary School Mathematics, a project funded by the U.S. Office of Education, 1970.

Leading teacher educators, such as Bloom (1968), Glaser (1968), and Lindval (1961), have questioned the adequacy of this means-referenced conception of instruction. They point out that a means-referenced orientation is inadequate for most instructional decision making. In its place, they have suggested that teachers begin to ask a different, more fundamental question--a question based on a goal-referenced instructional model. This model is illustrated in Figure 1.

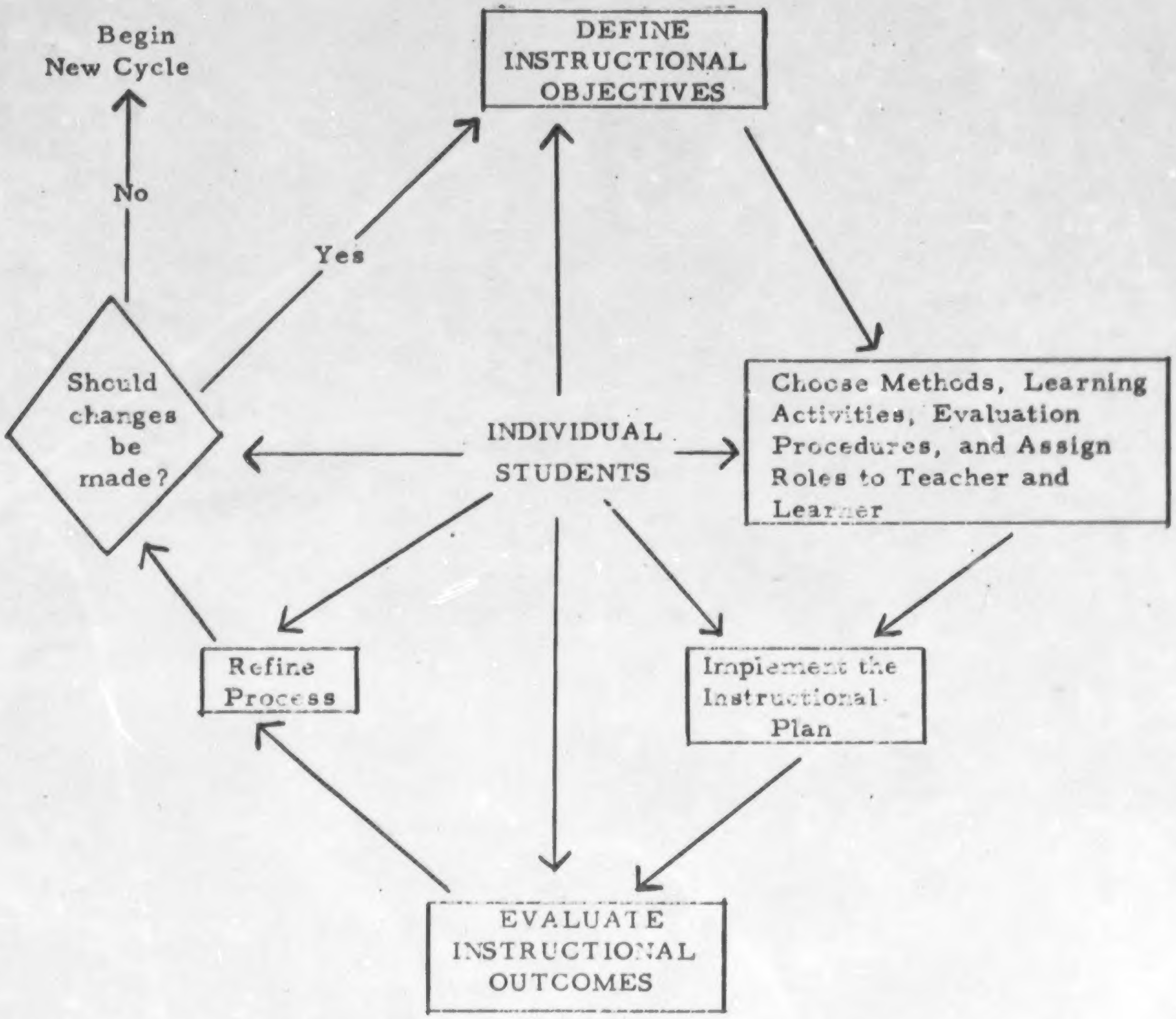
Begin
New Cycle
No
Yes
DEFINE INSTRUCTIONAL OBJECTIVES
Should changes be made?
INDIVIDUAL STUDENTS
Choose Methods, Learning Activities, Evaluation Procedures, and Assign Roles to Teacher and Learner
Refine Process
Implement the Instructional Plan
EVALUATE INSTRUCTIONAL OUTCOMES

Figure 1. A goal-referenced model for systematized instruction.

In general, the goal-referenced instructional model asks the teacher to give first priority to the goals and objectives of instruction. This implies that the question the teacher should ask first is "What do I want my learners to learn?" rather than

"What materials, techniques, and strategies should I use?" The essential point is that the goal-referenced model stresses the decision making the teacher engages in before and after instruction. This stress is accomplished by focusing the teacher's attention first on the role that objectives should play in planning and evaluating instruction rather than just by encouraging the teacher to use a particular set of materials without considering the outcomes it might produce.

EMERGING TYPES OF INDIVIDUALIZED INSTRUCTION

The use of the goal-referenced instructional model raises two basic issues for teachers who wish to individualize instruction: Who should determine the instructional objectives, and who should select the means to attain them? In reality, there are situations where the teacher is the primary determiner of both what and how students should be taught, and then, of course, there are situations where the student is the primary determiner. Experienced teachers know that such issues are not clear-cut, and this certainly is true for individualized instructional settings. Figure 2 presents a matrix identifying four popular types of individualized instruction and specifies the source from which statements of objectives, and means for achieving those objectives could originate. The writer's purpose in identifying and describing these four types of individualized instruction is not to narrow the concept of individualized instruction but rather to broaden it. He especially does not want to give the impression that a teacher should use one type of individualized instruction to the exclusion of the others. In fact, the essential point is that the selective use of each of the four types deserves serious consideration in order to meet the individual needs of students.

Source of Instructional Means	Source of Instructional objectives	
	Teacher	Learner
Teacher	A. Individually prescibed instruction	B. Personalized instruction
Learner	C. Self-directed instruction	D. Independent study

Figure 2. A matrix showing the source from which objectives and instructional means originate for four types of individualized instruction.

A Model for Using Diagnosis in Individualizing Mathematics.....

Individually Prescribed Instruction (IPI): Individually prescribed instruction does not refer only to the Individually Prescribed Instruction Project of the Learning Research and Development Center at the University of Pittsburgh. The basic point of view represented by type A assumes the needs of each learner to be unique. It also maintains that the teacher, as a trained professional, should identify the learner's individual needs and prescribe appropriate instruction designed to meet those needs. Type A, therefore, is characterized by the following:

1. Carefully developed pretests designed to diagnose the specific needs of the learner.

2. Clearly specified behavioral objectives that are formulated by the teacher, based on the results of diagnosis.

3. Carefully constructed methods and materials developed to enable the learner to reach each specified behavior.

4. Individually determined pacing which is a characteristic of all types of individualized instruction.

Personalized Instruction (PI): The basic strategy represented by type B was developed to provide the learner with opportunities to practice decision making so that he will see and feel the results of his decision. This type of individualization is characterized by the following:

1. The learner choosing and defining personal learning objectives; a learning contract is usually employed.

2. The teacher selecting the resources and instructional procedures to be used by the student in reaching his objectives.

3. The learner and teacher selecting prescribed evaluation criteria to be used to determine when the learner has fulfilled his contract.

Self-Directed Instruction (SDI): The basic assumption of those teachers employing type C is that each individual's strengths, learning strategies, and learning style are unique. From this assumption it follows logically that any preconceived sequence or system does the learner an injustice by denying him freedom to fully develop his individuality by prescribing learning activities that he should be learning to prescribe for himself. One of the reasons learners go to school is to learn how to be self-directed, self-actualizing people. Hence, the type C approach is characterized by the following:

1. A well-developed testing program that is used by the teacher to identify the learner's needs.

2. A set of clearly stated objectives that facilitate the teacher's attempt to com-

2. (continued).... municate these instructional objectives to the learner.

3. A well-developed learning laboratory with a wide variety of materials that can be placed at the disposal of the learner, since he is the one who chooses the instructional activities to achieve specified outcomes.

Independent Study (IS): The basic philosophy underlying type D asserts that the end result of all education is to develop learners who can clearly identify their own objectives, which are c[illegible]tent with their value system, and then choose appropriate means for achie[illegible]g these objectives. Therefore, type D is characterized by the following activi[illegible]s of the learners:

1. Determining the objectives they desire to pursue.

2. Selecting their own materials from a well-developed learning resource center.

3. Evaluating their progress in consultation with the teacher.

NEED FOR DIAGNOSIS

When one compares the four types of individualized instruction just described with the goal-referenced model in Figure 1, another important question arises: How crucial is diagnosis for teachers wishing to make effective use of a goal-referenced instructional model within the context suggested by the four types of individualized instruction?

As indicated in the introduction, the primary purpose of the individualized instruction is to change the typical group-paced classroom situation so that each student receives an appropriate set of instructional activities. This means an attempt should be made to increase motivation by identifying the learner's interests and providing learning experiences compatible with those interests and with the achievement of specified learning outcomes. When individualizing instruction, therefore, the teacher has a continuing need for information about each learner. The operation involved in obtaining essential data and background information about each learner and the process of analyzing that data and information is called diagnosis. Without the various types of diagnostic information, it is possible that teachers could waste instructional time teaching a student what he already knows or, as often occurs, frustrate the learner by attempting to develop behaviors and skills that he does not have the prerequisites or interests to successfully develop.

MODIFYING THE TEACHER'S CONCEPTION OF DIAGNOSIS

Current teacher-preparation programs and many mathematics-methods texts attempt to help teachers cope with readiness and sequencing of instruction by presenting a conception of diagnosis that is means oriented. That is, they focus primarily on the different types of diagnostic techniques and procedures that are available, how to administer and construct the various kinds of tests, and how to interpret the data obtained from standardized and teacher-made tests. All of these are important skills. This particular emphasis, however, leads the teacher to believe that the first and most important question to be considered before beginning an instructional unit is "What diagnostic techniques, instruments, or procedures should I use before and during my instructional unit?"

HOW ADEQUATE IS THIS MEANS-ORIENTED CONCEPTION OF DIAGNOSIS ?

There are at least two ways to test the adequacy of the means-oriented conception of diagnosis. The first is to determine whether the research in elementary school mathematics supports a means orientation. The second is to examine the requirements of the context within which the diagnostic information is to be utilized in order to see whether the means of obtaining diagnostic information should be the teacher's first or primary consideration.

Consider first the research on individualizing instruction in elementary school mathematics. A comprehensive survey of this research literature has been summarized by Suydam and Weaver (1970). How does diagnosis aid in individualizing instruction? Suydam and Weaver conclude that the research dealing with diagnosis shows that teachers should ascertain the specific errors that a pupil is making, determine specifically how he works, and give specific remedial help designed to decrease his errors. This research finding is clearly describing the type of individualized instruction referred to in Figure 2 as type A, or Individually Prescribed Instruction.

Consider next the requirements of the goal-referenced instructional model and its use in a IPI setting. Figure 1 and the characteristics of Individually Prescribed Instruction indicate that the first and most important requirement for using this instructional model in an IPI context would be the formulation of precise instructional objectives for each individual learner. This means that diagnosis should identify the pupil behaviors that should be used to formulate or select precise behavioral objectives for individual students.

A comparison of the research findings on individualizing mathematics instruction with the requirements of the goal-referenced instructional model as used in an IPI setting reveals a consistent pattern. The comparison seems to argue in favor of a conception of diagnosis that gives primary emphasis to the use the teacher makes of the results of diagnosis rather than just to his being concerned about the diagnostic means per se. That is, the comparison suggests that instead of asking a question concerning the means of diagnosis, the teacher should ask a question more directly related to the IPI setting--"What specific behaviors do my learners possess or lack at the beginning of an instructional unit?"

Such a goal-referenced conception of diagnosis and its relationship to the goal-referenced instructional model as used in an IPI context is presented in Figure 3.

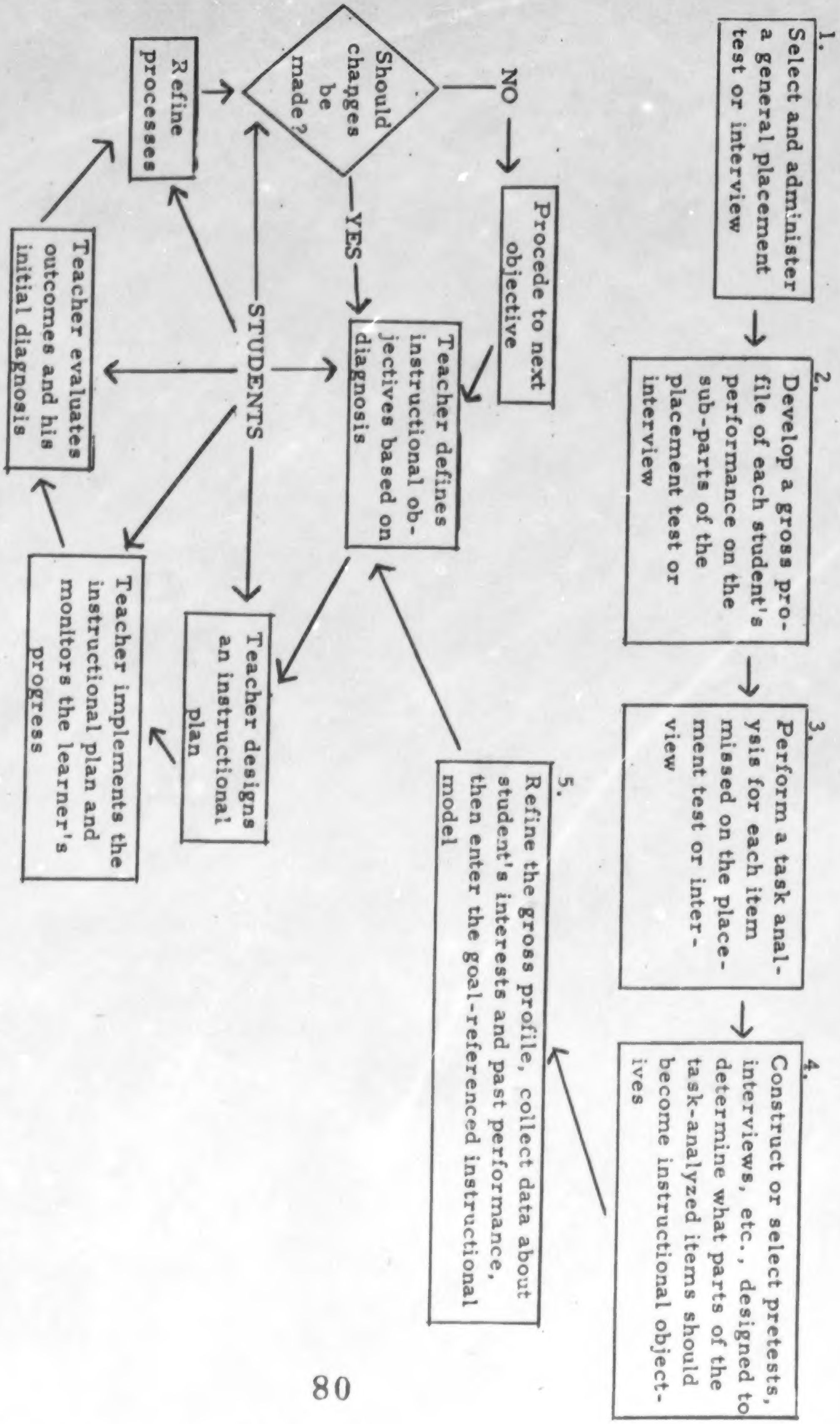
1. Select and administer a general placement test or interview
2. Develop a gross profile of each student's performance on the sub-parts of the placement test or interview
3. Perform a task analysis for each item missed on the placement test or interview
4. Construct or select pretests, interviews, etc., designed to determine what parts of the task-analyzed items should become instructional objectives
5. Refine the gross profile, collect data about student's interests and past performance, then enter the goal-referenced instructional model
Proceed to next objective
NO
Should changes be made?
YES
Refine processes
Teacher defines instructional objectives based on diagnosis
STUDENTS
Teacher designs an instructional plan
Teacher implements the instructional plan and monitors the learner's progress
Teacher evaluates outcomes and his initial diagnosis

FIGURE 3. A model for using goal-referenced diagnosis in the elementary school classroom to individualize instruction.

Essentially, the procedural model presented in Figure 3 indicates that the teacher should begin Individually Prescribed Instruction after diagnosing the learner's readiness or status in steps that lead to precisely defined instructional objectives. Designing an instructional plan suitable for each learner is not easy; it requires additional, careful planning by the teacher. Next, the model suggests that the teacher should continually monitor the learner's progress as he negotiates the instructional plan. The final steps of the model show that the accuracy and usefulness of the initial diagnosis should be evaluated in terms of how well it facilitated the teacher's choice of effective instructional activities.

To use the goal-referenced model of diagnosis in the context of the other three types of individualized instruction presented in Figure 2 would require some modification. First, depending on the type of individualized instruction involved, the student would be asked to assume some responsibility for defining learning outcomes, selecting instructional activities, and evaluating his own performance. Obviously, the amount and extent of this type of participation would depend on several factors, such as age, grade level, and the student's past experience with the various types of individualized instruction. Second, the learner would be required to do some self-diagnosis; therefore, the type of instructional objectives would differ in form and substance from the type of teacher-developed outcomes used in the IPI setting. Third, since the student would be asked to do self-evaluation, the evaluation procedures and the use made of the results would differ. For example, some self-reporting techniques might be developed by the teacher and used by the student to report his progress for an independent-study unit.

From this brief discussion, one can see that use of the goal-referenced model of diagnosis in the other three types of individualized instruction raises some interesting topics for further discussion, as well as some exciting possibilities for future curriculum development and research. It is hoped that the example given for the IPI context illustrates how diagnosis can be combined with the goal-referenced instructional model to individualize mathematics instruction in the elementary school classroom.

SUMMARY

This paper suggests that the current emphasis on the recognition of individual differences in learners has resulted in (1) a major change for the teacher, (2) a revision in the teacher's conception of instruction, (3) the emergence of four types of individualized instruction, and (4) a need to revise the teacher's conception of diagnosis from a means-referenced to a goal-referenced orientation. Finally, the writer presented a model for conducting diagnosis which might facilitate a teacher's attempt to use diagnosis for individualizing mathematics instruction in the elementary school classroom.

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